HW due Apr 4

Tao Ju

April 4, 2014

Problem 25
Prove that $[F_c : \mathbb{Q}]$ is infinite. F_c is the field of constructible reals (straight edge and compass).

Proof:
It is clear $\sqrt[2^n]{2} \in F_c$ for any n, which is a root of $x^{2^n} - 2 = 0$. Let $f(x) = x^{2^n} - 2$, then $f(x) \in \mathbb{Z}[x]$. Notice that $f(x)$ satisfies Eisenstein's Criterion for $p = 2$, as to be irreducible in $\mathbb{Z}[x]$. By Gauss's Lemma, $f(x)$ is irreducible in $\mathbb{Q}[x]$. By Thm 43, $[\mathbb{Q}[\sqrt[2^n]{2}] : \mathbb{Q}] = \deg(f) = 2^n$. Thus

ok

$$[F_c : \mathbb{Q}] = [F_c : \mathbb{Q}[\sqrt[2^n]{2}]] \cdot [\mathbb{Q}[\sqrt[2^n]{2}] : \mathbb{Q}] \geq 1 \cdot 2^n.$$

Therefore, $[F_c : \mathbb{Q}]$ is infinite.

Problem 26
Prove that if $2^m - 1$ is prime, then m is prime.

ok

Proof:
If $2^m - 1$ is prime, then $m \geq 2$. Suppose m is not a prime, then it can be decomposed into $m = pq$, where $p, q \geq 2$. Thus $2^m - 1 = 2^{pq} - 1 = (2^p)^q - 1 = (2^p - 1)((2^p)^{q-1} + (2^p)^{q-2} + \cdots + 1)$. The two factors are both greater than 2, so $2^m - 1$ is not prime, contradiction.
Therefore, m need to be a prime.

HW due Apr 7

Tao Ju

April 6, 2014

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Problem 27
Find the roots of

$$x^3 + 3x^2 + 6x + 5 = 0$$

using addition, subtraction, multiplication, division, and extraction of roots, i.e., solvability by radicals.

Solution:
Let $x = y - 1$, then

$$(y-1)^3 + 3(y-1)^2 + 6(y-1) + 5 = 0 \quad \Rightarrow \quad y^3 + 3y + 1 = 0.$$

Let $y = u - \frac{1}{u}$, then

$$(u - \frac{1}{u})^3 + 3(u - \frac{1}{u}) + 1 = 0 \quad \Rightarrow \quad (u^3)^2 + u^3 - 1 = 0$$

Which has roots $u^3 = \frac{-1 \pm \sqrt{5}}{2}$. We just assume $u^3 = \frac{-1+\sqrt{5}}{2}$, as to say

$$u_1 = \sqrt[3]{\frac{-1+\sqrt{5}}{2}}, \quad u_2 = e^{\frac{2\pi i}{3}} \cdot \sqrt[3]{\frac{-1+\sqrt{5}}{2}}, \quad u_3 = e^{-\frac{2\pi i}{3}} \cdot \sqrt[3]{\frac{-1+\sqrt{5}}{2}}$$

Back to $x = y - 1 = u - \frac{1}{u} - 1$, we obtain all the three roots:

$$x_1 = \sqrt[3]{\frac{-1+\sqrt{5}}{2}} - \sqrt[3]{\frac{1+\sqrt{5}}{2}} - 1.$$

$$x_2 = e^{\frac{2\pi i}{3}} \cdot \sqrt[3]{\frac{-1+\sqrt{5}}{2}} - e^{-\frac{2\pi i}{3}} \cdot \sqrt[3]{\frac{1+\sqrt{5}}{2}} - 1.$$

$$x_3 = e^{-\frac{2\pi i}{3}} \cdot \sqrt[3]{\frac{-1+\sqrt{5}}{2}} - e^{\frac{2\pi i}{3}} \cdot \sqrt[3]{\frac{1+\sqrt{5}}{2}} - 1.$$