Stephen Taylor

Math 542 3/3 HW

22 Prove for any prime p and positive integer n that p divides $\binom{p^n}{k}$ for any k with $0<k<p^n$.

Let F be a field of characteristic p. Then $q(x,y)=(x+y)^{p^n}\in F[x,y]$.

Notice $(x+y)^{p^n}=x^{p^n}+y^{p^n}$.

Proof by induction:

$n=1$ $\binom{p}{k}=\frac{p!}{k!(p-k)!}$ is an integer. p divides $p!$, but if $0<k<p$ then p does not divide $k!$ or $(p-k)!$, so p divides $\binom{p}{k}$.

Thus $(x+y)^p=\sum_{k=0}^{p}\binom{p}{k}x^{p-k}y^k=x^p+y^p$ since F has characteristic p.

If $(x+y)^{p^n}=x^{p^n}+y^{p^n}$, then $(x+y)^{p^{n+1}}=\left((x+y)^{p^n}\right)^p$
$=\left(x^{p^n}+y^{p^n}\right)^p$
$=x^{p^{n+1}}+y^{p^{n+1}}$

So by induction $(x+y)^{p^n}=x^{p^n}+y^{p^n}$.

Now $q(x,y)=(x+y)^{p^n}=\sum_{k=0}^{p^n}\binom{p^n}{k}x^{p^n-k}y^k=x^{p^n}+y^{p^n}$, so we can conclude $\binom{p^n}{k}=0$ in F for $0<k<p^n$, ie p divides $\binom{p^n}{k}$ for $0<k<p^n$.

Math 542
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22. Prove that $\forall p$ prime, $\forall n$, $\forall k$, $0<k<p^n$, $p \mid \binom{p^n}{k}$.

Let k s.t. $0<k<p^n$.

We have $\binom{p^n}{k} = \frac{p^n!}{k!(p^n-k)!} = \frac{p^n \cdot (p^n-1)!}{k \cdot (k-1)!(p^n-k)!} = \frac{p^n}{k} \cdot \binom{p^n-1}{k-1}$, so $k\binom{p^n}{k} = p^n\binom{p^n-1}{k-1}$. This means that $p^n \mid k\binom{p^n}{k}$. But $k<p^n$, so it must be the case that $p \mid \binom{p^n}{k}$. ∎