Stephen Taylor

Math 542 2/21 HW

17 Prove that $v_1, \dots, v_n$ are linearly dependent iff $v_1 = \vec{0}$ or $v_{i+1} \in \text{span}\{v_1, \dots, v_i\}$ for some $1 \le i < n$.

$\boxed{\Rightarrow}$ Let $v_1, \dots, v_n$ be linearly dependent. Then $\exists\ \alpha_1, \dots, \alpha_n$ not all 0 such that $\alpha_1 v_1 + \dots + \alpha_n v_n = \vec{0}$. Let $1 \le i \le n$ be the largest such that $\alpha_i \ne 0$. Then if $i = 1$, $\alpha_i v_1 = \vec{0}$ and $v_1 = \vec{0}$. If $i \ne 1$, then $-\alpha_i v_i = \alpha_1 v_1 + \dots + \alpha_{i-1} v_{i-1}$

$$v_i = \frac{\alpha_1}{-\alpha_i} v_1 + \dots + \frac{\alpha_{i-1}}{-\alpha_i} v_{i-1}$$

and $v_i \in \text{span}\{v_1, \dots, v_{i-1}\}$.

OK

$\boxed{\Leftarrow}$ If $v_1 = \vec{0}$ then the vectors are trivially linearly dependent. Suppose $v_{i+1} \in \text{span}\{v_1, \dots, v_i\}$ for some $1 \le i < n$. Then $\exists\ \alpha_1, \dots, \alpha_i$ such that

$$v_{i+1} = \alpha_1 v_1 + \dots + \alpha_i v_i$$
$$0 = \alpha_1 v_1 + \dots + \alpha_i v_i - v_{i+1}$$

The coefficient of v_{i+1} is not 0, so this shows $\{v_1, \dots, v_n\}$ are linearly dependent using the combination $\alpha_1 v_1 + \dots + \alpha_i v_i - v_{i+1} + 0 v_{i+2} + \dots + 0 v_n = 0$

This shows $v_1, \dots, v_n$ are linearly dependent iff $v_1 = \vec{0}$ or $v_{i+1} \in \text{span}\{v_1, \dots, v_i\}$ for some $1 \le i < n$.