Problem 12,13

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Problem 1 *(Fri Feb 14) Question (August J.) Suppose every subgroup of finite group G is a normal subgroup. Must G be abelian?*

Counterexample. Consider quaternion group

$$Q_8 = \langle x, y \mid x^2 = y^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1} \rangle.$$

It is easy to compute that the conjugate class of Q_8 is

OK

$$\{1\}, \{x^2\}, \{x, x^3\}, \{y, x^2 y\}, \{xy, x^3 y\}.$$

It follows that $Z(Q_8) = \{1, x^2\}$. By computation, we know Q_8 has only one 2-order element, so $Z(Q_8)$ is the only 2-order subgroup. Since $Z(Q_8)$ is normal, then each 2-order subgroup of Q_8 is normal. Let G be 4-order subgroup, then $[Q_8 : G] = 2$, which implies $Q_8 = G \cup gG = G \cup Gg$ for each $g \notin G$. So $gG = Gg$ for all $g \in Q_8$, and hence G is normal in Q_8. It is clear that the other subgroups of Q_8, namely 1 and Q_8, are normal too.

Problem 2 *(Fri Feb 14)*

(a) Suppose P is a p-Sylow subgroup of G and H a subgroup such that $P \lhd H$ and $H \lhd G$. Prove that $P \lhd G$.

(b) If $K \lhd H$ and $H \lhd G$, does it follow that $K \lhd G$? Show that the answer is No. Consider $G = S_4$, $K = \{id, \sigma\}$ where $\sigma = (12)(34)$ and $H = \{id, \sigma, \tau, \rho\}$ where τ and ρ are conjugates of σ. Determine what τ and ρ are and show that $K \lhd H$ and $H \lhd G$, but K is not a normal subgroup of G.

(a) *Proof.* Let gPg^{-1} be an arbitrary p-Sylow subgroup of G, where $g \in G$. Since $P \subset H$ and $H \lhd G$, $gPg^{-1} \subset H$ for all $g \in G$. Since $P \lhd H$, $gPg^{-1} \subset P$ for all $g \in G$, which implies P is a normal subgroup of G.

(b) *Proof.* Since $\sigma = (12)(34)$ has type of two 2-cycles, so without loss of generality we can write $\tau = (13)(24)$ and $\rho = (14)(23)$. It is easy to see H is a subgroup. Since H is disjoint union of two conjugate classes $\{e\}$ and $\{\sigma, \tau, \rho\}$, H is normal in G. Since $[H, K] = 2$, $K \lhd H$. OK

Since σ is conjugate to τ, ρ in G. It is obvious K is not normal in G.

Homework 10

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Problem 1 : *(Fri Feb 14) Question (August J.) Suppose every subgroup of finite group G is a normal subgroup. Must G be abelian?*

Problem 1 Solution: G can be not abelian. One of the examples is Quaternion group:

$$G = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$$

ok

12 Since $ij = k$ and $ji = -k$ it is obvious that G is not abelian.
Then it is easy to observe that it has four nontrivial subgroup, we need to check whether they are normal.

1. $H = \{1, -1\}$
   It is obviously normal because for any $g \in G$, $gHg^{-1} = H$ with $g1g^{-1} = 1$ and $g(-1)g^{-1} = -1$

2. $H = \{1, -1, i, -i\}$,$H = \{1, -1, j, -j\}$,$H = \{1, -1, k, -k\}$
   These three nontrivial subgroups are equivalent, so we just need to prove one of them are normal. For the first group $H = \{1, -1, i, -i\}$, since we have proved that $H = \{1, -1\}$ is normal, we just need to show that any conjugation act on $\{i. - i\}$ is in H. we begin by calculation:

$$\begin{aligned} jij^{-1} &= (-k)(-j) = kj = -i \\ kik^{-1} &= j(-k) = -jk = -i \\ j(-i)j^{-1} &= k(-j) = -kj = i \\ k(-i)k^{-1} &= (-j)(-k) = -jk = -i \end{aligned}$$

   So we can conclude that H is normal.

So G is an non-abelian group whose subgroups are all normal groups.

13 **Problem 2 :** *(Fri Feb 14)*
(a) Suppose P is a p-Sylow subgroup of G and H a subgroup such that $P \lhd H$ and $H \lhd G$. Prove that $P \lhd G$.
(b) If $K \lhd H$ and $H \lhd G$, does it follow that $K \lhd G$? Show that the answer is No. Consider $G = S_4$, $K = \{id, \sigma\}$ where $\sigma = (12)(34)$ and $H = \{id, \sigma, \tau, \rho\}$ where τ and ρ are conjugates of σ. Determine what τ and ρ are and show that $K \lhd H$ and $H \lhd G$, but K is not a normal subgroup of G.

Problem 2 Solution:

part (a): Suppose P is not a normal subgroup of G. Then there exists another p-Sylow subgroup P_1 and $g \in G$, such that $gPg^{-1} = P_1$. Since H is a normal subgroup of G, then we can conclude that $gPg^{-1} \subset gHg^{-1} = H$. Apparently P_1 is still a p-Sylow subgroup of G. Then we can know that P and P_1 are both p-Sylow subgroup of H. According to *Corollary 18* we can conclude that P is not a normal subgroup of H, where we get a contradiction. So P is a normal subgroup of G. Ok

part (b): First it is easy to notice that τ and ρ are conjugate of σ. Since it is the product of two two-cycles, we can conclude that $\tau = (13)(24)$ and $\rho = (14)(23)$.
K is obviously not normal because we can find an element $(123) \in S_4$ such that:

$$(123)(12)(34)(321) = (14)(23) \notin K$$

But H is normal in G because the conjugate of all elements are in H, which follow from the fact that σ, τ, ρ is a conjugate classes. And K is normal in H because

$$\begin{aligned} \tau\sigma\tau^{-1} &= (13)(24)(12)(34)(13)(24) = (12)(34) \\ \rho\sigma\rho^{-1} &= (14)(23)(12)(34)(14)(23) = (12)(34) \end{aligned}$$

So if $K \lhd H$ and $H \lhd G$, K may not be normal in G

Verify that H is a subgroup.