Math 542-Modern Algebra II

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OK

Problem:
(Mon Feb 10) Prove for any $n \geq 3$ that $Z(S_n) = \{id\}$.

Solution:
Let $\alpha \in S_n$ be choosen arbitrary such that $\alpha \neq e$ and set a, b such that $\alpha(a) = b$, where $a \neq b$. Then, let $\beta \in S_n$ such that β is the two cycle: $\beta = (b\,c)$, with $c \neq a$. We can find such a c since $n \geq 3$, and so β fixes a. Now, we can see that:

$$\beta\alpha\beta^{-1}(a) = \beta\alpha(a) = \beta(b) = c.$$

Wheras:

$$\alpha(a) = b.$$

Hence, $\beta\alpha\beta^{-1} \neq \alpha$, which shows that $\beta\alpha \neq \alpha\beta$, and hence no element in S_n commutes with every other element of S_n, other than $e \in S_n$. Hence, $Z(S_n) = e$.

Math 542

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10. *Prove for any $n \geq 3$ that $Z(S_n) = \{e\}$.*

    Let τ be an element of S_n not equal to e. Then τ has a cycle decomposition $c_1 \cdots c_k$. Since $n > 2$, there is always a way to create a distinct cycle decomposition $\rho = c'_1 \cdots c'_k$ with $|c_i| = |c'_i|$: Switch the numbers 1 and 2 in the original representation. This will preserve the lengths of all the cycles, while creating a distinct element, unless τ contains the 2-cycle (1 2). In that case, switch 1 and 3. Then $\tau \neq \rho$, but τ is conjugate to ρ by Proposition 15. Therefore $\tau \notin Z(S_n)$, so $Z(S_n) = \{e\}$.