$\frac{6}{6}$

Homework 7

Yuan Chaojie (Ernest)

February 14, 2014

Problem 1 :

1. *Suppose G is a finite abelian group which contains an element which has non-square-free order. Prove that for some prime p it has an element of order p^2.*
2. *Suppose a is an element of a finite abelian group G with order p^2 let $b = a^p$, let $H =< b >$ be the subgroup generated by b and suppose K is a subgroup of G with $K \cap H = \{e\}$. Prove that a is not an element of HK.*
3. *Suppose G_1, G_2 are finite abelian groups with $|G_1|$ and $|G_2|$ relatively prime. Show that for any subgroup $H \subseteq G_1 \times G_2$ there are subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ such that $H = H_1 \times H_2$. (Warning: the relatively prime hypothesis is necessary.)*
4. *Suppose G_1, G_2 are finite abelian groups with $|G_1|$ and $|G_2|$ relatively prime. Show that if G_1 and G_2 both have the CP then $G_1 \times G_2$ has CP.*[1]
5. *Prove that $C_p \times C_p \times \cdots \times C_p$ has the CP.*
6. *Prove Problem 4.*

Problem 1 Solution:

OK

1. Suppose claim 1 is true, then there exists an element $a \in G$,$a \neq e$ and a prime p, such that $|a| = p^2 n$. Then we look at the element a^n, by the definition of order we can know that $(a^n)^{p^2} = a^{np^2} = e$ and any positive integar less than p^2 can not be its order (otherwise the order of a will be smaller than $p^2 n$). So a^n is the elment we want.

OK

2. Suppose not in the claim two that a is an element of $H \times K$, then there exists $k \in K$ and positive integar l, such that $(a^p)^l k = a$, which means $k = a^{1-pl}$. Then using the fact that $|a| = p^2$:

$$k^p = a^{(1-pl)p} = a^{p-p^2 l} = a^p = b$$

then we can conclude that b is an element in $H \bigcap K$. So $b = e$, we get a contradiction. So a is not an element of HK.

[1] CP is defined after Problem 4.

OK

3. we prove it by constructing two mappings :ϕ_1 maps $H \to H_1$, where $\phi_1(h_1, h_2) = h_1$ and ϕ_2 maps $H \to H_2$, where $\phi_2(h_1, h_2) = h_2$. It is simple to notice that H_1 and H_2 is a subgroup of G_1 and G_2. Since it is similar we can just prove it for H_1. Let $g_1, g_2 \in H_1$, then there exists $(g_1, l_1) \in H$ and $(g_2, l_2) \in H$. Then $\phi((g_1, l_1)(g_2, l_2)) = \phi((g_1g_2, l_1l_2)) = g_1g_2$. So we proved that $g_1g_2 \in H_1$. Moreover since (e_1, e_2) is the identity in H, then we can conclude that e_1 is in H_1. And for any $(g_1, l_1) \in H$, its inverse $(g_1^{-1}, l_1^{-1}) \in H$, so $g_1^{-1} \in H_1$. So H_1 is indeed an subgroup of G_1. Similarly H_2 is indeed an subgroup of G_2.
   Then we will prove that $H_1 \times H_2 = H$. By the construction we can know that $H_1 \times H_2 \supset H$, it is sufficient to prove that $H_1 \times H_2 \subset H$. We choose any $h_1 \in H_1, h_2 \in H_2$, we need to prove that $(h_1, h_2) \in H$. Since $|G_1|$ and $|G_2|$ are relatively prime, then there exists s, t, such that $s|G_1| + t|G_2| = 1$. As $h_1 \in H_1$, there exists h_2', such that $(h_1, h_2') \in H$. Then
   $$(h_1, h_2')^{t|G_2|} = (h_1^{t|G_2|}, h_2'^{t|G_2|}) = (h_1^{1-s|G_1|}, e_2) = (h_1, e_2) \in H$$
   Similarly $(e_1, h_2) \in H$. So $(h_1, e_2)(e_1, h_2) = (h_1, h_2) \in H$. So we proved that $H_1 \times H_2 \subset H$. In conclusion : for any subgroup $H \subseteq G_1 \times G_2$ there are subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ such that $H = H_1 \times H_2$. ($|G_1|$ and $|G_2|$ relatively prime)

OK

4. Since $|G_1|$ and $|G_2|$ relatively prime,we can use the claim above. Suppose $G_1 \times G_2$ has CP, then we prove G_1 has CP(G_2 is similarly proved that has CP). Suppose H_1 is a subgroup of G_1, then we know that $H = H_1 \times \{e_2\}$ is a subgroup of G. Since G has CP then using the claim above, we can conclude that there exists $K = K_1 \times K_2$, such that $H \times K = G$. Then we claim that $H_1 \times K_1 = G_1$. Obviously $H_1 \times K_1 \subset G_1$. Then suppose there exists $g \in G$, such that $g \notin H_1 \times K_1$, then we can know that (g, e_2) can not be in $H \times K$, which contradicts with the fact that $H \times K = G_1 \times G_2$. So in conclusion G_1 has CP.
   Suppose G_1 and G_2 has CP. Since any subgroup of $G_1 \times G_2$ has the form $H = H_1 \times H_2$, then by the CP, we know there exists K_1, K_2 that satisfy the condition $H_1 \times K_1 = G_1$ and $H_2 \times K_2 = G_2$. Then we let $K = K_1 \times K_2$, we need to prove that K is a subgroup and $H \times K = G$. It is obviously a subgroup since K_1 and K_2 are both subgroups, then there product, identity and inverse will be in $K_1 \times K_2$. Then we notice that $H \times K \subset G$ because it is the product of two subgroups. It is sufficient to prove that $G \subset H \times K$. Let any $(g_1, g_2) \in G$. Since $H_1 \times K_1 = G_1$, then there exists h_1, k_1 such that $h_1k_1 = g_1$. Similarly, there exists h_2, k_2 such that $h_2k_2 = g_2$. So $(g_1, g_2) = (h_1, h_2)(k_1, k_2) \in H \times K$. So in conclusion: $G = G_1 \times G_2$ has CP.

OK

5. Suppose each cyclic group are generated by q_i, where $i = 1, 2..., n$. Suppose K is a subgroup of G_1. Then $K \simeq C_{p_1}^k$, where $C_{p_1}^k = C_{p_1} \times C_{p_1} \times \ldots \times C_{p_1}$ for k times. Then let K' to be the subgroup of G_1 while K' has the maximum order among the subgroups which intersects K are just $\{e\}$. Claim that $K \cdot K'$ is just G_1. Otherwise, suppose $K \cdot K'$ is just a proper subgroup of G_1, then there exists a subgourp $\langle q_i \rangle$, which satisfy the condition that it is not in $K \cdot K'$. Since $\langle q_i \rangle$ has only trivial groups, so $\langle q_i \rangle \bigcap K \cdot K' = \{e\}$, which implies that $K \bigcap (K' \cdot \langle q_i \rangle) = \emptyset$. But it condicts with the fact that K' is the subgroup of G has the maximum order which intersects G equals $\{e\}$ ($K' \cdot \langle q_i \rangle$ is a subgroup of G is a trivial fact.) So In conclusion:$K \cdot K' = G_1$.

Ok

6. By the first two claim we can know that if G has CP but exists an element has non-square-free order, we will get a contradiction by the second claim. So each element is square-free.
Then Since G is finite abelian, we could use the structure theorem to decompose it into p-groups. Since G_i can be expressed as a product of cyclic groups and every element of G has square-free order, we could get:

$$G_i \simeq C_p \times C_p \times \ldots \times C_p$$

By the previous claim we can know that G_i has CP. And then by claim 4 we can conclude that G has CP. So $2 \to 1$ is proved.