Math 542 Exercises 7,8

7 - ok
8 - ✓

Joe Timmerman

February 12, 2014

7

False. As a counterexample, take $G_1 = G_2 = \mathbb{Z}_4 \times \mathbb{Z}_2, H_1 = \langle 2 \rangle \times \{0\}, H_2 = \{0\} \times \mathbb{Z}_2$. Note that $H_1 = \{(2,0),(0,0)\}$, so $H_1 \cong H_2$ by $(2,0) \mapsto (0,1), (0,0) \mapsto (0,0)$. Now consider $G_1/H_1 = (\mathbb{Z}_4 \times \mathbb{Z}_2)/(\langle 2 \rangle \times \{0\}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G_2/H_2 = (\mathbb{Z}_4 \times \mathbb{Z}_2)/(\{0\} \times \mathbb{Z}_2) \cong \mathbb{Z}_4 \times \{0\} \cong \mathbb{Z}_4$. Note that for any prime p, $Z_{p^2} \ncong \mathbb{Z}_p \times \mathbb{Z}_p$, because p is not relatively prime with itself (and so $\mathbb{Z}_p \times \mathbb{Z}_p$ is not cyclic, while $\mathbb{Z}_{p^2}$ is). Thus, we have that $G_1 \cong G_2, H_1 \cong H_2$, but $G_1/H_1 \ncong G_2/H_2$, so the statement is not true.

8

Prove that $\mathrm{Stab}(ga) = g\,\mathrm{Stab}(a)g^{-1}$.
Assume $G \curvearrowright X$, and let $a \in X$.

Claim 0.1.

$$\mathrm{Stab}(ga) \subseteq g\,\mathrm{Stab}(a)g^{-1}$$

Proof. Let $h \in \mathrm{Stab}(ga)$. Then by definition of the stabilizer group, $h(ga) = ga$. Using the definition of a group action, this means that $(hg)a = ga$. Multiplying on the left by g^{-1}, we have $(g^{-1}hg)a = a$, which means $(g^{-1}hg) \in \mathrm{Stab}(a)$. Then multiplying on left by g and on the right by g^{-1}, we get that $h \in g\,\mathrm{Stab}(a)g^{-1}$. Thus, every element of $\mathrm{Stab}(ga)$ is also in $g\,\mathrm{Stab}(a)g^{-1}$, so we have inclusion as desired. □

Claim 0.2.

$$g\,\mathrm{Stab}(a)g^{-1} \subseteq \mathrm{Stab}(ga)$$

Proof. Let $h \in g\,\mathrm{Stab}(a)g^{-1}$. Some for some $x \in \mathrm{Stab}(a)$, $h = gxg^{1-} \implies hg = gx$. Since $x \in \mathrm{Stab}(a)$, we have that $(gx)a = g(xa) = ga$. Then, using $gx = hx$, this becomes $(hg)a = ga \implies g(ga) = ga$. Thus, $h \in \mathrm{Stab}(ga)$, and $\mathrm{Stab}(a)g^{-1} \subseteq \mathrm{Stab}(ga)$.

□

I've shown mutual inclusion, so $\mathrm{Stab}(ga) = g\,\mathrm{Stab}(a)g^{-1}$.