5 ψ 2 YUE XU

Prob 6 Suppose G_1, G_2, H_1, H_2 are finite abelian groups, $G_1 \times G_2 \simeq H_1 \times H_2$, $G_1 \simeq H_1$ Prove $G_2 \simeq H_2$

Give a counter-example if the word finite is dropped.

Pf. Since G_1, G_2, H_1, H_2 are finite abelian group they can be written as products of cyclic p-groups

$$G_1 \simeq H_1 \simeq C_{p_1^{n_{11}}} \times C_{p_1^{n_{12}}} \times \cdots \times C_{p_1^{n_{1k_1}}} \times \cancel{C_{p_1} \times \cdots \times C_{p_1}}$$
$$\times C_{p_2^{n_{21}}} \times C_{p_2^{n_{22}}} \times \cdots \times C_{p_2^{n_{2k_2}}} \times \cancel{C_{p_2} \times \cdots \times C_{p_2}}$$
$$\times \cdots \times C_{p_m^{n_{m1}}} \times \cdots \times C_{p_m^{n_{mk_m}}} \times \cancel{C_{p_m} \times \cdots \times C_{p_m}}$$

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(if $n_{ij} = 0$, then delete $C_{p_1^{n_{ij}}}$ from the product)

where $n_{ij} \geq 0$ $n_{i1} \geq n_{i2} \geq \cdots n_{ik_i}$ $i = 1, 2, \cdots m$

$$G_2 \simeq C_{p_1^{\tilde{n}_{11}}} \times C_{p_1^{\tilde{n}_{12}}} \times \cdots \times C_{p_1^{\tilde{n}_{1k_1}}} \times \cancel{C_{p_1} \times \cdots \times C_{p_1}} \times \cdots \times C_{p_m^{\tilde{n}_{m1}}} \times \cdots \times C_{p_m^{\tilde{n}_{mk_m}}} \times \cancel{C_{p_m} \times \cdots \times C_{p_m}}$$

$$H_2 \simeq C_{p_1^{\hat{n}_{11}}} \times C_{p_1^{\hat{n}_{12}}} \times \cdots \times C_{p_1^{\hat{n}_{1k_1}}} \times \cancel{C_{p_1} \times \cdots \times C_{p_1}} \times \cdots \times C_{p_m^{\hat{n}_{m1}}} \times \cdots \times C_{p_m^{\hat{n}_{mk_m}}} \times \cancel{C_{p_m} \times \cdots \times C_{p_m}}$$

where $\tilde{n}_{ij}, \hat{n}_{ij} \geq 0$ $\tilde{n}_{i1} \geq \tilde{n}_{i2} \geq \cdots \geq \tilde{n}_{ik_i}$ $\hat{n}_{i1} \geq \cdots \geq \hat{n}_{ik_i}$ $i = 1, \cdots m$

Then $G_1 \times G_2 \simeq C_{p_1^{n_{11}}} \times \cdots \times C_{p_1^{n_{1k_1}}} \times C_{p_1^{\tilde{n}_{11}}} \times \cdots \times C_{p_1^{\tilde{n}_{1k_1}}} \times \cdots$
$$\times C_{p_m^{n_{11}}} \times \cdots \times C_{p_m^{n_{1k_1}}} \times C_{p_m^{\tilde{n}_{11}}} \times \cdots \times C_{p_m^{\tilde{n}_{mk_m}}}$$

$$H_1 \times H_2 \simeq C_{p_1^{n_{11}}} \times \cdots \times C_{p_1^{n_{1k_1}}} \times C_{p_1^{\hat{n}_{11}}} \times \cdots \times C_{p_1^{\hat{n}_{1k_1}}} \times \cdots \times C_{p_m^{n_{11}}} \times \cdots \times C_{p_m^{n_{1k_1}}} \times C_{p_m^{\hat{n}_{11}}} \times \cdots \times C_{p_m^{\hat{n}_{mk_m}}}$$

Since $G_1 \times G_2 \simeq H_1 \times H_2$,

According to the rule of uniqueness,

$\tilde{n}_{11} = \hat{n}_{11}, \cdots \tilde{n}_{1k_1} = \hat{n}_{1k_1} \cdots \tilde{n}_{ij} = \hat{n}_{ij}$ $i = 1, \cdots m, j = 1, \cdots, k_i$

Thus H_2, G_2 are the product of same cyclic groups

so $G_2 \simeq H_2$

Counter example:

let $G_1 \simeq H_1 \simeq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots$ infinite product of $\mathbb{Z}$

$G_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}$ $H_2 \simeq \mathbb{Z}_2$

$G_1 \times G_2 \simeq \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z} \times \cdots$

$H_1 \times H_2 \simeq \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z} \times \cdots$

So $G_1 \times G_2 \simeq H_1 \times H_2$

$G_1 \simeq H_1$ but $G_2 \not\simeq H_2$

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