Ian Campbell

3) Let $|G| = n = p_1^{a_1} \cdot \dots \cdot p_k^{a_k}$

Suppose n is square-free.

OK

Then $a_1 = a_2 = \cdots = a_k = 1$.

Thus, by corollary 4, $G \simeq G_1 \times \cdots \times G_k$ where $|G_i| = p_i$.

Since each p_i is prime, every G_i is cyclic.

Since each G_i is cyclic, they are unique up to isomorphism, thus there can only be one such G up to isomorphism.

Let $|G| = n = p_1^{a_1} \cdot \dots \cdot p_k^{a_k}$

Now suppose that G is isomorphic to every other abelian group of order n.

Further suppose that some $a_i > 1$.

Then $\mathbb{Z}_{p_i} \times \mathbb{Z}_{\frac{n}{p_i}}$ is an abelian group of order n.

However, since p_i and $\frac{n}{p_i}$ are not relatively prime, $\mathbb{Z}_n \not\simeq \mathbb{Z}_{a_i} \times \mathbb{Z}_{\frac{n}{a_i}}$ by theorem 2.

Since $\mathbb{Z}_n$ and $\mathbb{Z}_{p_i} \times \mathbb{Z}_{\frac{n}{p_i}}$ are not isomorphic, G cannot be isomorphic to both of them.

But by supposition G is isomorphic to every abelian group of order n.

We conclude that there cannot be an $a_i > 1$.

So each $a_i = 1$.

So n is square-free.