Stephen Tingle

Math 542 1/24 HW

1 – ok
2 – ok

1 a) Find $x \in \mathbb{Z}$ such that $x = 6 \mod 10$ and

ok $x = 15 \mod 21$ and $0 \le x \le 210$.

By inspection $1 = 1\cdot 21 - 2\cdot 10$. Following the method we used to prove the Chinese Remainder Theorem, $x_0 = 6\cdot 21 - 15\cdot 2\cdot 10 = -174$ should work, although it is not in the right range. $210 = 0$ both mod 10 and 21, so $x = x_0 + 210 = 36$ should also work. Check:

$$36 = 3\cdot 10 + 6$$
$$36 = 1\cdot 21 + 15$$
$$0 \le 36 \le 210$$

$$\boxed{x = 36}$$

b) Find the smallest positive integer y

ok such that $y = 6 \mod 10$, $y = 15 \mod 21$, and $y = 8 \mod 11$.

From part a, $y = 6 \mod 10$ and $y = 15 \mod 21$ if $y = 36 \mod 210$, so I will reduce this problem to:

$$\begin{cases} y = 36 \mod 210 \\ y = 8 \mod 11. \end{cases} \qquad \begin{aligned} 210 &= 19\cdot 11 + 1 \\ 210 - 19\cdot 11 &= 1 \end{aligned}$$

Repeating the process from part a, $y_0 = 8\cdot 1\cdot 210 - 36\cdot 19\cdot 11 = -5844$

should work. $10 = 2\cdot 5$, $21 = 3\cdot 7$, and 11 are pairwise coprime, so all solutions to the problem will be congruent mod $10\cdot 21\cdot 11 = 2310$. One solution is -5844, so the set of all solutions is $\{-5844 + n\cdot 2310 \mid n \in \mathbb{Z}\}$. The smallest positive integer y in this set is $\boxed{1086 = y}$

Check: $1086 = 108\cdot 10 + 6$
$1086 = 51\cdot 21 + 15$
$1086 = 98\cdot 11 + 8$

2 a) Find integers i, j such that there is no integer x with $x \equiv i \bmod 6$ and $x \equiv j \bmod 15$.

ok

$i=0, j=1$. are two such numbers. This follows from:

<u>Lemma</u> For $i, j, n, m \in \mathbb{Z}$ with $n, m > 0$ $\exists$ x such that $x \equiv i \bmod n$ and $x \equiv j \bmod m$ iff $g = \gcd(n, m)$ divides $j - i$.

<u>Proof</u> $\exists$ x such that

$$\begin{cases} x \equiv i \bmod n \\ x \equiv j \bmod m \end{cases} \Longleftrightarrow \exists\, \alpha, \beta \in \mathbb{Z} \text{ such that } \begin{cases} x = i + \alpha n \\ x = j + \beta m \end{cases}$$

$$\Longleftrightarrow \quad i + \alpha n = j + \beta m$$

$$\alpha n - \beta m = j - i$$

$\Leftrightarrow$ $g | j - i$ by Bezout's Theorem $\square$

Here $g = \gcd(6, 15) = 3$, so since 3 does not divide $1 - 0 = 1$, there is no such x.

b) Find all pairs i, j with $i \in \{0, \dots, 5\}$ and $j \in \{0, \dots, 14\}$ such that there is an integer x with $x \equiv i \bmod 6$ and $x \equiv j \bmod 15$.

ok

By the lemma, these are $\{i, j \mid 3 | j - i;\ i, j \in \mathbb{Z},\ 0 \leq i \leq 5,\ 0 \leq j \leq 14\}$

Specifically:

i	j	i	j	i	j
0	0	2	2	4	1
0	3	2	5	4	4
0	6	2	8	4	7
0	9	2	11	4	10
0	12	2	14	4	13
1	1	3	0	5	2
1	4	3	3	5	5
1	7	3	6	5	8
1	10	3	9	5	11
1	13	3	12	5	14

Ziyuan Tao

1-ok

2-ok

542 Homework. Due Jan 31. 2014

①. Find $x \in \mathbb{Z}$

ok

$$\begin{cases} x = 6 \mod 10 \\ x = 15 \mod 21 \end{cases}$$

$0 \le x \le 210$

Solution: Assume $\begin{cases} x = 10k_1 + 6 \\ x = 21k_2 + 15 \end{cases}$ $k_1, k_2 \in \mathbb{Z}$

Since $0 \le x \le 210$, we have $0 \le k_1 \le 20$, $0 \le k_2 \le 9$

$10k_1 + 6 = 21k_2 + 15$

$10k_1 = 21k_2 + 9$

$10 \mid (21k_2 + 9)$.

So k_2 can only be 1 or 11.

If $k_2 = 1$, $k_1 = 3$.

If $k_2 = 11$, $k_1 = 24 > 20$

So $k_1 = 3$ and $k_2 = 1$ is the only possible solution.

$x = 10 \times 3 + 6 = 36$

②. Find smallest positive y such that

ok

$$\begin{cases} y = 8 \mod 11 & (1) \\ y = 6 \mod 10 & (2) \\ y = 15 \mod 21. & (3) \end{cases}$$

Solution.

<u>Proposition</u>: If x also solves these equations, then $x = y \pmod{10 \times 21 \times 11}$.

<u>Proof</u>: for simplicity,

let $a_1 = 8$ $n_1 = 11$

$a_2 = 6$ $n_2 = 10$

$a_3 = 15$. $n_3 = 21$.

for $k = 1, 2, 3$, $x = y = a_k \pmod{n_k}$.

So $(x - y) = (a_k - a_k) = 0 \pmod{n_k}$.

$n_k \mid (x - y)$.

Since n_1, n_2, n_3 are pairwise relatively prime,

$n_1 n_2 n_3 \mid (x-y)$

So $x \equiv y \pmod{n_1 n_2 n_3}$ ▨

So there is only one solution between

0 and $n_1 n_2 n_3 - 1$ (if there is one).

We know that y solves the system in (a).

So $y \equiv 36 \pmod{10 \times 21 = 210}$

$y \equiv 8 \pmod{11}$.

$8 + 11k_2 = 210 + 36k_1$

One solution is $\begin{cases} k_1 = 5 \\ k_2 = 98. \end{cases}$

then $y = 8 + 11 \times 98 = 1086$.

$10 \times 11 \times 21 > 2000$.

So $\underline{\underline{1086}}$ is the only positive solution smaller than

$10 \times 11 \times 21$, hence it is the smallest solution

Problem 2.

Ok

(a). Find integers i, j such that there is no integer x

with $x \equiv i \bmod 6$ and $x \equiv j \bmod 15$

(b). Find all pairs (i, j) with $i = 0, \ldots, 5$, $j = 0, \ldots, 14$

such that there is an integer x with $x \equiv i \bmod 6$

and $x \equiv j \bmod 15$.

<u>Solution:</u> there are only 90 possible pairs. So

we can simply ennumerate them to see if a given

pair (i, j) has a corresponding x that satisfies $x \equiv i \bmod 6$

$x \equiv j \bmod 15$.

If such an x exists, then there must be one between 0 and 89 incl -si

because: If $x \equiv i \bmod 6$ $\quad 0 \le i \le 5$

$x \equiv j \bmod 15$ $\quad 0 \le j \le 14$

If we let $y =$ the remainder of x when divided by 90

then $0 \le y \le 89$. $\quad x = 90k + y$. $\therefore y = x - 90k$

$y = x - 90k \equiv i \pmod 6$

$y = x - 90k \equiv j \pmod{15}$.

So y solves the same system and we only need to search for $0 \le y \le 89$ to see if it's a solution.

Problem (2) continued.

We can solve this problem using computer.

the source code is provided here. On the next page is a detailed solution, which answers both part (a) and part (b).

homework2.cpp

```
#include <stdio.h>

int main()
{
  FILE *fout=fopen("log.out","w");
  int i,j,k,x,y,z;
  bool yes=false;

    for(j=0; j<6; ++j)
      for(k=0; k<15; ++k)
      {
        x=j; y=k;
        yes=false;
        for(i=1; i<=90; ++i)
        {
          if(i%6==j && i%15==k)
           {
            yes=true;
            z=i;
            fprintf(fout,"(%d,%d): x=%d\n",x,y,z);
            break;
           }
        }
        if(!yes)
          fprintf(fout,"(%d,%d): no such x exists\n",x,y);
      }
    return 0;
}
```

```
(0,0): x=30
(0,1): no such x exists
(0,2): no such x exists
(0,3): x=18
(0,4): no such x exists
(0,5): no such x exists
(0,6): x=6
(0,7): no such x exists
(0,8): no such x exists
(0,9): x=24
(0,10): no such x exists
(0,11): no such x exists
(0,12): x=12
(0,13): no such x exists
(0,14): no such x exists
(1,0): no such x exists
(1,1): x=1
(1,2): no such x exists
(1,3): no such x exists
(1,4): x=19
(1,5): no such x exists
(1,6): no such x exists
(1,7): x=7
(1,8): no such x exists
(1,9): no such x exists
(1,10): x=25
(1,11): no such x exists
(1,12): no such x exists
(1,13): x=13
(1,14): no such x exists
(2,0): no such x exists
(2,1): no such x exists
(2,2): x=2
(2,3): no such x exists
(2,4): no such x exists
(2,5): x=20
(2,6): no such x exists
(2,7): no such x exists
(2,8): x=8
(2,9): no such x exists
(2,10): no such x exists
(2,11): x=26
(2,12): no such x exists
(2,13): no such x exists
(2,14): x=14
(3,0): x=15
(3,1): no such x exists
(3,2): no such x exists
(3,3): x=3
(3,4): no such x exists
(3,5): no such x exists
(3,6): x=21
(3,7): no such x exists
(3,8): no such x exists
(3,9): x=9
(3,10): no such x exists
(3,11): no such x exists
(3,12): x=27
(3,13): no such x exists
(3,14): no such x exists
(4,0): no such x exists
(4,1): x=16
(4,2): no such x exists
```

```
(4,3): no such x exists
(4,4): x=4
(4,5): no such x exists
(4,6): no such x exists
(4,7): x=22
(4,8): no such x exists
(4,9): no such x exists
(4,10): x=10
(4,11): no such x exists
(4,12): no such x exists
(4,13): x=28
(4,14): no such x exists
(5,0): no such x exists
(5,1): no such x exists
(5,2): x=17
(5,3): no such x exists
(5,4): no such x exists
(5,5): x=5
(5,6): no such x exists
(5,7): no such x exists
(5,8): x=23
(5,9): no such x exists
(5,10): no such x exists
(5,11): x=11
(5,12): no such x exists
(5,13): no such x exists
(5,14): x=29
```