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Vector Spaces

A vector space, V, is a set with two operations, vector addition (written $u + v$) and scalar multiplication (written av). The elements of V will be denoted using u, v, w, etc. The formula '$u \in V$' is short hand for 'u is an element of V' or 'u in V' or just 'u is a vector'. Vector spaces will be written using capital letters V, W, etc. Scalars are elements of some field F, for example, the real numbers, $\mathbb{R}$, or the complex numbers, $\mathbb{C}$. Scalars will be written using the letters a, b, c, etc.

Closure axioms:

1. If $u \in V$ and $v \in V$, then $u + v \in V$.
2. If $u \in V$ and a a scalar, then $au \in V$.

Associative, commutative, distributive axioms:

1. For all $u, v, w \in V$ $\quad (u + v) + w = u + (v + w)$.
2. For all $u, v \in V$ $\quad u + v = v + u$.
3. For all scalars a and b and vectors $u \in V$ $\quad (ab)u = a(bu)$.
4. For all scalars a and b and vectors $u \in V$ $\quad (a + b)u = au + bu$.
5. For all scalars a and vectors $u, v \in V$ $\quad a(u + v) = au + av$.

Zero vector, additive inverse, identity axioms:

1. There exists a vector $\vec{0} \in V$ such that for all $u \in V$ $\quad \vec{0}+u = u+\vec{0} = u$.
2. For every $u \in V$ there exists a vector $v \in V$ (for which we write $v = -u$) such that $u + v = v + u = \vec{0}$.
3. For every $u \in V$, $\quad 1u = u$.

Any abstract set V with two operations, vector addition and scalar multiplication which satisfy all the above axioms is a vector space.

Most author's use either $\mathbf{0}$ or $\vec{0}$ to denote the zero vector. Note that it is not the same as the zero element 0 of the field.

Exercise 1 *Prove that* $0u = \vec{0}$ *for any* $u \in V$ *a vector space.*

Exercise 2 *Prove that* $(-1)u = -u$ *for any* $u \in V$ *a vector space.*

Definition 3 For W a subset of a vector space V (written $W \subseteq V$) we say that W is a <u>*subspace*</u> **of V iff**

1. **for every $u, v \in V$ if $u \in W$ and $v \in W$, then $u + v \in W$, and**
2. **for every $u \in V$ and scalar a if $u \in W$, then $au \in W$.**

Theorem 4 *If W is a subspace of V, then W is itself a vector space under the operations defined in V.*

proof:

The closure axioms are easy since they are practically the same as the definition of subspace. The associative, commutative, distributive axioms are true in W because they are true in V and W is a subset of V. The zero vector $\vec{0}$ is in W because $0u = \vec{0}$ (exercise 1) so (assuming W is nonempty) if anything is in W, then $\vec{0}$ is in W. Similarly $(-1)u = -u$ (exercise 2), so if $u \in W$, then also $-u \in W$. $\Box$

Theorem 5 *Suppose W is a subset of V (i.e., $W \subseteq V$). Then*

1. *W is a subspace of V*

   iff

2. *for every $u, v \in W$ and scalars a, b we have $au + bv \in W$.*

proof:

(1) implies (2):

Assume W is a subspace of V. Suppose $u, v \in W$ and a, b are scalars. By the second axiom of subspaces we have that $au \in W$ and $bv \in W$. Letting $w_1 = au$ and $w_2 = bv$ we have that $w_1 \in W$ and $w_2 \in W$, therefore by the first axiom of subspaces we have that $w_1 + w_2 \in W$ and so $au + bv \in W$.

(2) implies (1):

Assume (2): for every $u, v \in W$ and scalars a, b we have $au + bv \in W$. We must show the two axioms of a subspace hold for W. Suppose $u, v \in W$. Then letting $a = b = 1$ we have that $1u + 1v \in W$, so $1u + 1v = u + v \in W$. For the second axiom, suppose $u \in W$ and a any scalar, then we have that $au + 0u \in W$ by (2), but $au = au + \vec{0} = au + 0u$ so $au \in W$. $\Box$

Definition 6 For $u_1, \ldots, u_n$ elements of a vector space V, define their <u>span</u>:

$$\text{span}(\{u_1, u_2, \ldots, u_n\}) = \{a_1u_1 + a_2u_2 + \cdots + a_nu_n : a_1, a_2, ..., a_n \textbf{ scalars}\}$$

Each of the vectors $a_1u_1 + a_2u_2 + \cdots + a_nu_n$ is called a linear combination of the u's so we could also say that the span is the set of all linear combinations. If $W = \text{span}(\{u_1, u_2, \ldots, u_n\})$, we say that '$W$ is spanned by $u_1, u_2, \ldots, u_n$' or '$u_1, u_2, \ldots, u_n$ span W'. The closure axioms of a vector space V guarantee that if $u_1, u_2, \ldots, u_n \in V$, then $\text{span}(\{u_1, u_2, \ldots, u_n\}) \subseteq V$. This is true because the second closure axiom says each a_iu_i is in V, while the first axiom guarantees that their sum is in V, e.g., if we write $v_i = a_iu_i$, $v_1, v_2 \in V$ implies $v_1 + v_2 \in V$ and so $v_1 + v_2, v_3 \in V$ implies $v_1 + v_2 + v_3 = (v_1 + v_2) + v_3 \in V$, and so on.

Theorem 7 *Suppose $u_1, u_2, \ldots, u_n$ are elements of W which is a subspace of V. Then* $\text{span}(\{u_1, u_2, \ldots, u_n\}) \subseteq W$.

proof:

Suppose $v \in \text{span}(\{u_1, u_2, \ldots, u_n\})$. Then for some scalars, $a_1, \ldots, a_n$ we have that

$$v = a_1u_1 + \cdots + a_nu_n.$$

Since W is a subspace of V we have that $a_iu_i \in W$ for each i. Now let $v_i = a_iu_i$ to simplify our writing. Since $v_1 \in W$ and $v_2 \in W$ we have by the first axiom of subspaces that $v_1 + v_2 \in W$. Thus we have that the two vectors $v_1 + v_2$ and v_3 are elements of W. This means their sum $(v_1 + v_2) + v_3$ is in W. Continuing on like this we see that $v_1 + v_2 + \cdots + v_k \in W$ for each k and so

$$v = a_1u_1 + \cdots + a_nu_n = v_1 + v_2 + \cdots + v_n \in W$$

as we needed to show.$\Box$

Theorem 8 *Suppose* $u_1, u_2, \ldots, u_n$ *are elements of a vector space* V*. Then* $\mathrm{span}(\{u_1, u_2, \ldots, u_n\})$ *is a subspace of* V*.*

proof:

We verify each of the axioms of a subspace. Let

$$W = \mathrm{span}(\{u_1, u_2, \ldots, u_n\}).$$

Suppose v, w are elements of W. Then since W is the span of the $u's$ there exists scalars $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$ such that

$$v = c_1u_1 + \cdots + c_nu_n \text{ and } w = d_1u_1 + \cdots + d_nu_n.$$

But then

$$v + w = (c_1 + d_1)u_1 + (c_2 + d_2)u_2 + \cdots + (c_n + d_n)u_n$$

and so $v + u \in \mathrm{span}(\{u_1, u_2, \ldots, u_n\}) = W$

For the second axiom, suppose $v \in W$ and a a scalar. Then for some scalars $c_1, \ldots, c_n$

$$v = c_1u_1 + \cdots + c_nu_n$$

but then

$$av = a(c_1u_1 + \cdots + c_nu_n) = (ac_1)u_1 + \cdots + (ac_n)u_n$$

and so $av \in \mathrm{span}(\{u_1, u_2, \ldots, u_n\}) = W$.

□

It follows from these last two theorems that $\mathrm{span}(\{u_1, u_2, \ldots, u_n\})$ is the smallest subspace of V which contains the vectors $u_1, u_2, \ldots, u_n$.

Theorem 9 *Suppose* $u \in \mathrm{span}(\{u_1, \ldots, u_n\})$ *then*

$$\mathrm{span}(\{u, u_1, \ldots, u_n\}) = \mathrm{span}(\{u_1, \ldots, u_n\})$$

proof:

To show two sets A and B are equal, $A = B$, show that $A \subseteq B$ and $B \subseteq A$.

To show that $A \subseteq B$ suppose that $x \in A$ and then show $x \in B$.

First we show $\text{span}(\{u, u_1, \ldots, u_n\}) \subseteq \text{span}(\{u_1, \ldots, u_n\})$. Since $u \in \text{span}(\{u_1, \ldots, u_n\})$ there exists scalars $b_1, \ldots, b_n$ so that $u = b_1u_1 + \cdots + b_nu_n$. Now let w be any element of $\text{span}(\{u, u_1, \ldots, u_n\})$. This means there are scalars $a, a_1, \ldots, a_n$ such that

$$w = au + a_1u_1 + \cdots + a_nu_n.$$

But then substituting for u:

$$w = a(b_1u_1 + \cdots + b_nu_n) + a_1u_1 + \cdots + a_nu_n$$

$$= (ab_1 + a_1)u_1 + \cdots + (ab_n + a_n)u_n$$

and so $w \in \text{span}(\{u_1, \ldots, u_n\})$ as was to be shown.

Second we show $\text{span}(\{u_1, \ldots, u_n\}) \subseteq \text{span}(\{u, u_1, \ldots, u_n\})$. This is easier. Suppose $w \in \text{span}(\{u_1, \ldots, u_n\})$. Then there are scalars $c_1, \ldots, c_n$ so that

$$w = c_1u_1 + \cdots + c_nu_n$$

but then

$$w = 0u + c_1u_1 + \cdots + c_nu_n$$

so $w \in \text{span}(\{u, u_1, \ldots, u_n\})$ as was to be shown. $\Box$

Definition 10 **For $v_1, v_2, \ldots, v_n$ vectors in a vector space V we say that they are <u>linearly independent</u> iff for any scalars $a_1, a_2, \ldots, a_n$**

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = \vec{0} \to a_1 = a_2 = \cdots = a_n = 0$$

Definition 11 **We say $v_1, v_2, \ldots, v_n$ are <u>linearly dependent</u> iff $v_1, \ldots, v_n$ are not linearly independent.**

Definition 12 **$v_1, v_2, \ldots, v_n$ is a <u>basis</u> for the vector space V iff**

1. **$v_1, v_2, \ldots, v_n$ are linearly independent and**

2. **$V = \text{span}(\{v_1, v_2, \ldots, v_n\})$.**

Theorem 13 *Let A be any $n \times n$ matrix. Then A is invertible iff the set of columns of A is a basis for $\mathbb{R}^n$, i.e., $\mathrm{col}_1(A), \mathrm{col}_2(A), \ldots, \mathrm{col}_n(A)$ is a basis for $\mathbb{R}^n$.*

proof:

Before proving this result we first prove the following Lemma.

Lemma 14 *Suppose A is an $m \times n$ matrix and*

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

is $n \times 1$. Then

$$AB = A \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = b_1\mathrm{col}_1(A) + b_2\mathrm{col}_2(A) + \cdots + b_n\mathrm{col}_n(A) = \sum_{k=1}^{n} b_k\mathrm{col}_k(A)$$

proof:

Write the matrix A as follows:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & & & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{bmatrix}$$

i.e., $a_{i,j} = \mathrm{entry}_{i,j}(A)$. Then

$$AB = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & & & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1}b_1 + a_{1,2}b_2 + a_{1,3}b_3 + \cdots + a_{1,n}b_n \\ a_{2,1}b_1 + a_{2,2}b_2 + a_{2,3}b_3 + \cdots + a_{2,n}b_n \\ a_{3,1}b_1 + a_{3,2}b_2 + a_{3,3}b_3 + \cdots + a_{3,n}b_n \\ \vdots \\ a_{m,1}b_1 + a_{m,2}b_2 + a_{m,3}b_3 + \cdots + a_{m,n}b_n \end{bmatrix}_{m\times 1}$$

$$= b_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \\ \vdots \\ a_{m,1} \end{bmatrix} + b_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \\ \vdots \\ a_{m,2} \end{bmatrix} + b_3 \begin{bmatrix} a_{1,3} \\ a_{2,3} \\ a_{3,3} \\ \vdots \\ a_{m,3} \end{bmatrix} + \cdots + b_n \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ a_{3,n} \\ \vdots \\ a_{m,n} \end{bmatrix}$$

This proves the Lemma. □Now to prove the Theorem we first assume that A is invertible and show that the columns of A are a basis for $\mathbb{R}^n$. To see that they are independent, suppose that

$$b_1\text{col}_1(A) + \cdots + b_n\text{col}_n(A) = Z$$

where $Z_{n\times 1}$ is the zero vector. By the lemma

$$AB = Z$$

where B is the column vector made from $b_1, \ldots, b_n$. Since A is invertible we have that $B = A^{-1}Z = Z$ so $B = Z$ and so $b_i = 0$ for all i with $1 \le i \le n$. This shows the the columns of A are linearly independent. To see that they span $\mathbb{R}^n$, let C be an arbitrary element of $\mathbb{R}^n$ so that

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Since A is invertible if we set $B = A^{-1}C$ then we know that $AB = C$ and by the lemma we have that

$$b_1 \text{col}_1(A) + \cdots + b_n \text{col}_n(A) = C$$

and so C is in the span of the columns of A. This shows that if A is invertible then its columns are a basis. Next we prove the converse, using the contrapositive.

The contrapositive of the implication: P implies Q is (Not Q) implies (Not P) They are logically equivalent.

Assume that A is not invertible. Then as was shown the algorithm for attempting to invert A produces a $B_{n \times 1} \neq Z$ such that $AB = Z$. This means by the Lemma that

$$b_1 \text{col}_1(A) + \cdots + b_n \text{col}_n(A) = Z$$

and since $B \neq Z$ at least one $b_i \neq 0$. But this means that the columns of A are linearly dependent and hence not a basis. This finishes the proof of the Theorem. □

Theorem 15 *$v_1, v_2, \ldots, v_n$ are linearly dependent iff there are scalars*

$$a_1, a_2, \ldots, a_n$$

such that $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = \vec{0}$ and for at least one i we have $a_i \neq 0$.

proof:

Not For ALL x Statement(x) is logically equivalent to There exists x such that Not Statement(x)

So negating linear independence gives us:
There exists scalars $a_1, \ldots, a_n$ such that
Not [$a_1 v_1 + \cdots + a_n = \vec{0}$ implies $a_1 = a_2 = \cdots = a_n = 0$].

The negation of an implication Not [P implies Q] is logically equivalent to P and Not Q.

So in this case we get
$a_1v_1 + \cdots + a_n = \vec{0}$ and Not $[a_1 = a_2 = \cdots = a_n = 0]$.

But Not $[a_1 = a_2 = \cdots = a_n = 0]$ is the same as saying 'at least one of the a_i is not equal to 0. $\Box$

Exercise 16 *Prove that $v_1, \ldots, v_n$ are linearly dependent iff $v_1 = \vec{0}$ or $v_j \in \mathrm{span}(\{v_1, \ldots, v_{j-1})$ for some j with $1 < j \leq n$.*

Lemma 17 *(Exchange) Suppose $v_1, v_2, \ldots, v_{k+1}, w_1, \ldots, w_m$ are vectors in a vector space V and*

1. *$v_1, v_2, \ldots, v_{k+1}$ are linearly independent, and*
2. *$\mathrm{span}(\{v_1, v_2, \ldots, v_k, w_1, \ldots, w_m\}) = V$.*

Then for some i with $1 \leq i \leq m$

$$\mathrm{span}(\{v_1, v_2, \ldots, v_{k+1}, w_1, w_2 \ldots, w_{i-1}, w_{i+1}, \ldots w_m\}) = V$$

(i.e., we have added v_{k+1} and removed w_i.)

proof:

By (2) there are scalars a_i, b_j such that

$$v_{k+1} = a_1v_1 + a_2v_2 + \cdots + a_kv_k + b_1w_1 + b_2w_2 + \cdots + b_mw_m.$$

It must be that for some i with $1 \leq i \leq m$ that $b_i \neq 0$, because otherwise we would have that

$$v_{k+1} = a_1v_1 + a_2v_2 + \cdots + a_kv_k$$

and therefore

$$\vec{0} = a_1v_1 + a_2v_2 + \cdots + a_kv_k + (-1)v_{k+1}$$

contradicting their independence (1).

Therefore we have that

$$\begin{aligned} -b_iw_i \quad = & \quad a_1v_1 + \cdots + a_kv_k + (-1)v_{k+1} + \\ + & \quad b_1w_1 + \cdots + b_{i-1}w_{i-1} + b_{i+1}w_{i+1} + \cdots + b_mw_m \end{aligned}$$

and since $b_i \neq 0$

$$\begin{aligned} w_i \;=\;& \tfrac{a_1}{-b_i}v_1 + \cdots + \tfrac{a_k}{-b_i}v_k + \tfrac{(-1)}{-b_i}v_{k+1} + \\ +\;& \tfrac{b_1}{-b_i}w_1 + \cdots + \tfrac{b_{i-1}}{-b_i}w_{i-1} + \tfrac{b_{i+1}}{-b_i}w_{i+1} + \cdots + \tfrac{b_m}{-b_i}w_m. \end{aligned}$$

Hence

$$w_i \in \text{span}(\{v_1, \ldots, v_{k+1}, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_m\})$$

and so by Theorem 9 we have

$$\text{span}(\{v_1, v_2, \ldots, v_{k+1}, w_1, w_2 \ldots, w_{i-1}, w_{i+1}, \ldots w_m\})$$

$$= \text{span}(\{v_1, v_2, \ldots, v_{k+1}, w_1, w_2 \ldots, w_m\})$$

and by (2) we have

$$\text{span}(\{v_1, v_2, \ldots, v_{k+1}, w_1, w_2 \ldots, w_m\}) = V$$

and so we are done. □

Theorem 18 *(Main Theorem) If a vector space V can be spanned by n vectors, then any set of $n+1$ vectors in V is linearly dependent.*

proof:

Suppose for contradiction that $\text{span}(\{u_1, \ldots, u_n\}) = V$ and $v_1, \ldots, v_{n+1}$ are linearly independent.

Step 1. Apply the Exchange Lemma with $k = 0$ to obtain i so that $\text{span}(\{v_1, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots u_n\}) = V$.

Step 2. Rename (relabel? reorder?) the u's so that

$$(w_1, \ldots, w_{n-1}) = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots u_n)$$

and apply the Exchange Lemma with $k = 1$ to obtain i so that

$$\text{span}(\{v_1, v_2, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots w_{n-1}\}) = V.$$

Step k. Given $\{w_1, \ldots, w_{n-k}\} \subseteq \{u_1, \ldots, u_n\}$ such that

$$\text{span}(\{v_1, v_2, \ldots, v_k, w_1, \ldots, w_{n-k}\}) = V,$$

apply the Exchange Lemma to find i so that

$$\text{span}(\{v_1, v_2, \dots, v_k, v_{k+1}, w_1, \dots, w_{i-1}, w_{i+1}, \dots w_{n-k}\}) = V$$

Last Step: Given that $\text{span}(\{v_1, v_2, \dots, v_{n-1}, w_1\}) = V$, apply the Exchange Lemma to get that

$$\text{span}(\{v_1, v_2, \dots, v_n\}) = V.$$

But this is a contradiction, since $v_{n+1} \in \text{span}(\{v_1, v_2, \dots, v_n\})$ implies that

$$v_{n+1} = a_1 v_1 + \cdots + a_n v_n$$

for some scalars a_i, but then

$$\vec{0} = (-1)v_{n+1} + a_1 v_1 + \cdots + a_n v_n$$

and therefore $v_1, \dots, v_{n+1}$ would be linearly dependent, a contradiction. □

Definition 19 The dimension of a vector space V is n, written $\dim(V) = n$ iff V has a basis of size n.

Theorem 20 *Any two bases for a vector space V have the same size.*

proof:
Otherwise, if $u_1, \dots, u_m$ is a basis for V and $v_1, \dots, v_n$ is another basis for V and $m < n$, then since the u's span V it must be that the v's are linearly dependent (by 18), contradicting that they are a basis. □

Theorem 21 *If V is a vector space, $\dim(V) = n$, and $u_1, \dots, u_n \in V$ are linearly independent, then $u_1, \dots, u_n$ are a basis for V.*

proof:
It is enough to show $\text{span}(\{u_1, \dots, u_n\}) = V$. By the main theorem (18) for any $v \in V$ we know that $u_1, u_2, \dots, u_n, v$ are linearly dependent. Hence there are scalars $a_1, a_2, \dots, a_n, a$ (at least one of which is nonzero) such that

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n + av = \vec{0}.$$

Since $u_1, \dots, u_n$ are linearly independent, it cannot be that $a = 0$ and so

$$v = -\frac{1}{a}(a_1 u_1 + a_2 u_2 + \cdots + a_n u_n)$$

and so $v \in \text{span}(\{u_1, \dots, u_n\})$ and since v was an arbitrary element of V we have that $\text{span}(\{u_1, \dots, u_n\}) = V$.□

Theorem 22 *Suppose* $\dim(V) = n$ *and*

$$\text{span}(\{u_1, ..., u_n\}) = V.$$

Then $u_1, ..., u_n$ *is a basis for* V*.*

proof:

It is enough to prove that $u_1, ..., u_n$ are linearly independent.

In a proof by contradiction assume the negation of what you are trying to prove and then reason to a contradiction. It follows logically that what you are trying to prove must be true

Suppose for contradiction that they are linearly dependent. Then there are scalars $a_1, \ldots, a_n$ such that

$$a_1 u_1 + \cdots + a_n u_n = \vec{0}$$

and for some i we have $a_i \neq 0$. But then we have that

$$u_i \in \text{span}(\{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n\})$$

and so by Theorem 9, we have that V is spanned by $n - 1$ vectors. This would imply by the main theorem (18) that any set of n vectors is linearly dependent, contradicting the fact the dimension of V is n.□

Theorem 23 *Every vector space has a basis.*

proof:

This theorem is true in general but requires a more sophisticated proof for the infinite dimensional case. Here we prove it just for the case that our vector space W is a subspace of a vector space V with finite dimension. Suppose the dimension of V is n.

If $W = \{\vec{0}\}$, then the dimension of W is 0 and the empty set is a basis for it. Otherwise let $v_1 \in W$ be an arbitrary vector in W not equal to the zero vector, $\vec{0}$. If v_1 spans W, then the dimension of W is 1 and v_1 is a basis for it. Otherwise choose any $v_2 \in W$ such that $v_2 \notin \text{span}\{v_1\}$. Continue this procedure. That is given $v_1, \ldots, v_k \in W$, if $v_1, \ldots, v_k$ span W, then stop. Otherwise choose $v_{k+1} \in W$ arbitrary but not in the span of $v_1, \ldots, v_k$. By an exercise 16 $v_1, \ldots, v_k$ are linearly independent for every k. By the main theorem $k \leq n$ so this process must stop after $\leq n$ steps and when it stops we have found a basis for W. □

Exercise 24 *Prove:*

If $\dim(V) = n$ *and* $u_1, \ldots, u_m \in V$ *are linearly independent, then we can extend this sequence to a basis of* V*. That is, we can find* $u_{m+1}, u_{m+2}, \ldots, u_n$ *so that* $u_1, \ldots, u_n$ *is a basis for* V*.*

Exercise 25 *If* $u_1, \ldots, u_n$ *span a vector space* V*, then there exists*

$$\{v_1, \ldots, v_m\} \subseteq \{u_1, \ldots, u_n\}$$

such that $v_1, \ldots, v_m$ *is a basis for* V*. In other words, any spanning set contains a basis.*

Theorem 26 *If* W *is a subspace of* V *and* $\dim(V) = n$*, then* $\dim(W) = m$ *for some* $m \leq n$*. If* $m = n$ *then* $W = V$*.*

proof:

Any basis for W is a set of m linearly independent vectors in V. But the main theorem (18) implies that $m \leq n$. Suppose $m = n$, then let $u_1, \ldots, u_n$ be a basis for W. Since they are linearly independent vectors in V, by Theorem 21 they must also be a basis for V, and so

$$W = \text{span}(\{u_1, \ldots, u_n\}) = V.$$

□