

Instructions

Do not hand in this sheet. Show all work. Write clearly and carefully. If you need extra time to rewrite your solution you will be given it. If you are in doubt as to whether something should be proved either ask me or supply a proof.

1. Define

- (a) n divides m .
- (b) p is a prime number.
- (c) $a \equiv_n b$ where a, b, n are integers and $n > 1$.
- (d) $(G, \cdot)$ is a group.
- (e) the order of $a \in G$, $|a|$.
- (f) σ is a cycle for σ some permutation
- (g) $\phi : G \rightarrow G'$ is a homomorphism (groups)
- (h) $\phi : G \rightarrow G'$ is an isomorphism (groups)
- (i) H is a normal subgroup of G

2. Prove **one** of the following:

- (a) (Euclid's Lemma) Suppose a, b, p are positive integers, p is a prime, and p divides ab . Prove that p divides a or p divides b .
- (b) Lagrange's Theorem: the order of a subgroup divides the order of the group.

3. Define

- (a) $(R, \cdot, +, 0)$ is a ring
- (b) commutative ring with 1
- (c) integral domain
- (d) field
- (e) ideal $I \subseteq R$

- (f) prime ideal
- (g) maximal ideal
- (h) $F[x]$ where F is a field
- (i) $\deg(f)$ where $f \in F[x]$
- (j) f is an irreducible polynomial in $R[x]$ for R a commutative ring

4. Give an example of

- (a) a ring which is not commutative.
- (b) a ring which is commutative but does not have a 1.
- (c) a commutative ring with 1 which is not an integral domain.
- (d) an integral domain which is not a field.
- (e) a finite field.
- (f) a prime ideal which is not maximal.
- (g) a polynomial in $\mathbb{Z}[x]$ which is irreducible in $\mathbb{Q}[x]$ but reducible in $\mathbb{R}[x]$.
- (h) a polynomial in $\mathbb{Z}[x]$ which is irreducible in $\mathbb{R}[x]$ but reducible in $\mathbb{C}[x]$.

5. Prove: Let R be a ring. Then for every $a \in R$

$$a0 = 0 = 0a$$

6. Prove **one** of the following:

- (a) Every finite integral domain is a field.
- (b) If R is a commutative ring with 1 and I is an ideal, then R/I is a field iff I is a maximal ideal.
- (c) If f, g are primitive polynomials in $\mathbb{Z}[x]$, then fg is primitive.
- (d) If $f \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Z}[x]$, then f is irreducible in $\mathbb{Q}[x]$. (You may use part (c) without proof)

7. Suppose $f(x) \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Z}[x]$ and $\alpha \in \mathbb{C}$ is one of its roots, i.e., $f(\alpha) = 0$. Suppose that $g(x) \in \mathbb{Q}[x]$ satisfies that $g(\alpha) = 0$. Prove that $\deg(g) \geq \deg(f)$.