

Instructions

Do not hand in this sheet. Show all work. Write clearly and carefully. If you need extra time to rewrite your solution you will be given it. If you are in doubt as to whether something should be proved either ask me or supply a proof. The two extra credit problems are optional.

1.

- (a) Define $f : X \rightarrow Y$ is one-to-one.
- (b) Define $f : X \rightarrow Y$ is onto.
- (c) Define g is the inverse of f .
- (d) Define transposition
- (e) Define n-cycle

2. Prove: Every cycle is the product of transpositions.

3. How many elements of order 4 are there there in S_5 ?

4.

- (a) Define $\phi : G \rightarrow G'$ is a homomorphism
- (b) Define isomorphism
- (c) Define automorphism
- (d) Define G is isomorphic to G'

5. Suppose $\phi : G \rightarrow G'$ is a homomorphism. Prove:

- (a) $\phi(e) = e'$.
- (b) for all $x \in G$ $\phi(x^{-1}) = (\phi(x))^{-1}$.

6. Let H be a subgroup of G and $a, b \in G$.
 - (a) Define aH
 - (b) Define Ha
 - (c) Define $aHbH$
 - (d) Define $H \triangleleft G$
7. State and Prove Lagrange's Thm. (Give all proofs, don't quote theorems or lemmas from the book.)
8. Let $\alpha : G \rightarrow G$ be an automorphism. Prove that α^{-1} is an automorphism of G .
9. Define $G \times H$ (this is the same as the book denotes $G \oplus H$).
10. Prove that $S_3 \times S_4$ is not isomorphic to a subgroup of S_5 .
11. Prove that $S_3 \times S_4$ is isomorphic to a subgroup of S_7 .
12. What is the order of 14 in the group $\mathbb{Z}_{48}$?
 What is the order of the coset $14 + 6\mathbb{Z}_{48}$ in the group $\mathbb{Z}_{48}/6\mathbb{Z}_{48}$?
 (Note: $6\mathbb{Z}_{48}$ is the subgroup of $\mathbb{Z}_{48}$ generated by 6.)
13. Prove: Suppose $\rho : G \rightarrow G'$ is any onto homomorphism and $H = \ker(\rho)$. Let $\phi : G \rightarrow G/H$ be the quotient map. Then there exists an isomorphism

$$\alpha : G/H \rightarrow G'$$
 such that $\rho = \alpha \circ \phi$.
14. (Extra Credit) Prove that $S_3 \times S_4$ is not isomorphic to a subgroup of S_6 .
15. (Extra Credit) Prove that the automorphism group of the rationals under addition is isomorphic to the multiplicative group of nonzero rationals.