

Instructions

Do not hand in this sheet. Show all work. Write clearly and carefully. If you need extra time to rewrite your solution you will be given it. If you are in doubt as to whether something should be proved either ask me or supply a proof.

1. Suppose p, n, m are integers.
 - (a) Define n divides m .
 - (b) Define p is a prime number.
 - (c) Define $\gcd(n, m)$.
 - (d) Define n and m are relatively prime.
2. (Euclid's Lemma) Suppose a, b, p are positive integers, p is a prime, and p divides ab . Prove that p divides a or p divides b .

You may quote other theorems which were proved before this, but you may not use the Prime Factorization Theorem since this Lemma was used to prove it.

3. Define $(G, *)$ is a group.
4. Suppose $(G, \cdot)$ is a group and $a \in G$.
 - (a) Define G is an abelian group.
 - (b) Define the order of G , $|G|$.
 - (c) Define the order of a , $|a|$.
 - (d) Define the subgroup generated by a , $\langle a \rangle$.
 - (e) Define the center of G , $Z(G)$.
 - (f) Define the centralizer of a , $C(a)$.
 - (g) Define b is the inverse of a , $b = a^{-1}$.
5. Let $(G, \cdot)$ be any group. Prove that for any $a, b \in G$,

$$(ab)^{-1} = b^{-1}a^{-1}$$

6. Let $(G, \cdot)$ be any group. Suppose that for every a and b in G that $a^2b^2 = (ab)^2$. Prove that G is abelian.

7. State the two (three?) step method for showing something is a subgroup by finishing the sentence “ H is a subgroup of G if and only if ...”

8. Let $(G, \cdot)$ be any group and a and b elements of G such that $ab = ba$. Define

$$H = \{x \in G : axb = bxa\}$$

Prove that H is a subgroup of G using the two step method.

9. Prove that any finite group with more than one element contains an element of prime order.

10. Define $\approx$ is an equivalence relation on the set X .

11. Define $a \equiv_n b$ where a, b, n are integers and $n > 1$.

12. Suppose $a, b, a', b', n \in \mathbb{Z}$ and $n > 1$. Prove that $a \equiv_n a'$ and $b \equiv_n b'$ implies that $ab \equiv_n a'b'$.

13. Prove that 7 divides $3^{3n} + 1$ for all positive odd integers n .

14. Define Z_n and $\oplus_n$ where n a positive integer.

15. Draw the lattice of subgroups of $(Z_{12}, \oplus_{12})$.

16. Define the group U_n . (The set and operation.)

17. Write out the Cayley table for U_{12} .

18. (Extra Credit Optional Problems)

(a) Suppose G has a unique element of order 2. Prove it must be in the center of G .

(b) Prove that U_{2^n} is not cyclic.

(c) Let a, b be elements of some group and suppose $aba^{-1} = b^{-1}$ and a has odd order. Prove that $b^2 = e$.

(d) If you learned the proof of some particular Theorem and are disappointed that I didn't ask for it, then here is your chance.

State and Prove (insert your choice here).