

Define P is a partition of the rectangle $R = [a, b] \times [c, d]$.

Define $L(P, f)$

Define $U(P, f)$

Lemma. If P_1 is a partition which refines P_2 , then

$$L(P_2, f) \leq L(P_1, f) \leq U(P_1, f) \leq U(P_2, f)$$

Lemma. If P_1 and P_2 are any two partitions, then

$$L(P_1, f) \leq U(P_2, f)$$

Define L and U

Define $f : R \rightarrow \mathbb{R}$ is Riemann integrable (Darboux definition).

Define $\iint_R f$

Example. Give an example of a bounded function f on $[0, 1] \times [0, 1]$ which is not Riemann integrable.

Theorem. If $f : R \rightarrow \mathbb{R}$ is continuous, then f is Riemann integrable.

Define $B \subseteq \mathbb{R}^2$ has zero area.

Theorem. If f is bounded on R and continuous except for a set of points with zero area, then f is Riemann integrable.

Define For $D \subseteq R$ an arbitrary region, define $\iint_D f$.

Define Boundary of D and interior of D .

Corollary. If $f : D \rightarrow \mathbb{R}$ is continuous and bounded on the bounded set $D \subseteq \mathbb{R}$ and the boundary of D has zero area, then the Riemann integral $\iint_D f$ exists.

Define For $f : [a, b] \rightarrow \mathbb{R}$ define f is Riemann integrable and $\int_a^b f$.

Theorem. (Fund Thm I.) Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous and define $F(x) = \int_a^x f$ for $x \in [a, b]$. Then F is continuous on $[a, b]$, differentiable on (a, b) and for every $x \in (a, b)$ $F'(x) = f(x)$.

Theorem. (Fund Thm II.) Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous and F is continuous on $[a, b]$, differentiable on (a, b) and for every $x \in (a, b)$ $F'(x) = f(x)$. Then $\int_a^b f = F(b) - F(a)$.

Theorem. (Substitution) Suppose f is continuous on $\phi([a, b])$ where ϕ is C' on some open set containing $[a, b]$. Then

$$\int_a^b f(\phi(t))\phi'(t)dt = \int_{\phi(a)}^{\phi(b)} f(x)dx.$$

Theorem. (Fubini) Suppose $f : R \rightarrow \mathbb{R}$ is continuous on the rectangle $R = [a, b] \times [c, d]$ and define $F : [c, d] \rightarrow \mathbb{R}$ by $F(x) = \int_c^d f(x, y)dy$. Then F is Riemann integrable and

$$\int_a^b F(x)dx = \iint_R f.$$

Similarly if $G(y) = \int_a^b f(x, y)dx$, then G is Riemann integrable and

$$\int_c^d G(y)dy = \iint_R f.$$

Hence,

$$\int_a^b (\int_c^d f(x, y)dy)dx = \int_c^d (\int_a^b f(x, y)dx)dy.$$

Corollary. (to the proof) In Fubini's Theorem instead of assuming f is continuous it is enough to assume $\iint_R f$ exists and for all $x \in [a, b]$ the integral $\int_c^d f(x, y)dy$ exists. Similarly, for the similarly.

Corollary. Suppose $\phi_1 : [a, b] \rightarrow \mathbb{R}$ and $\phi_2 : [a, b] \rightarrow \mathbb{R}$ are continuous with $\phi_1(x) \leq \phi_2(x)$ for all $x \in [a, b]$. Let

$$D = \{(x, y) : a \leq x \leq b \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}$$

and let $f : D \rightarrow \mathbb{R}$ be continuous. Then

$$\iint_D f = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy \right) dx.$$

Lemma. Suppose $g : D \rightarrow \mathbb{R}$ is continuous on the open set $D \subseteq \mathbb{R}^2$ and for every rectangle $R \subseteq D$

$$\iint_R g = 0.$$

Then $g(p) = 0$ for every $p \in D$.

Theorem. Suppose $f : D \rightarrow \mathbb{R}$ is C'' on the open set $D \subseteq \mathbb{R}^2$. Then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

at every point in D .

Theorem. (Liebnitz) Suppose that f is C' on an open set of the plane containing $(a, b) \times [c, d]$. Then

$$\frac{\partial}{\partial x} \int_c^d f(x, y) dy = \int_c^d \frac{\partial}{\partial x} f(x, y) dy$$

for every $x \in (a, b)$.

Define $\sum a_n$ converges.

Lemma. $\lim_{n \rightarrow \infty} x^n = 0$ if $0 \leq x < 1$ and $\lim_{n \rightarrow \infty} x^n = \infty$ if $x > 1$.

Theorem. $\sum_{n=0}^{\infty} x^n$ converges iff $|x| < 1$. It equals $\frac{1}{1-x}$ when it converges.

Theorem. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem. If $\sum a_n$ converges and $\sum b_n$ converges, then $\sum (a_n + b_n)$ converges and equals $\sum a_n + \sum b_n$.

Theorem. If $\sum a_n$ converges and $c \in \mathbb{R}$, then $\sum ca_n$ converges and equals $c \sum a_n$.

Theorem. (Cauchy Criterion) $\sum a_n$ converges iff $\forall \epsilon > 0 \exists N \forall n > m > N$

$$\left| \sum_{k=m}^n a_k \right| = |a_m + a_{m+1} + \cdots + a_n| < \epsilon.$$

Theorem. $\sum a_n$ converges iff for all N the tail $\sum_{n=N}^{\infty} a_n$ converges iff there exists N such that $\sum_{n=N}^{\infty} a_n$ converges.

Theorem. $\sum a_n$ converges implies $\lim_{n \rightarrow \infty} a_n = 0$.

Theorem. $\sum |a_n|$ converges implies $\sum a_n$ converges.

Define Absolute convergence.

Define Conditional convergence.

Theorem. (Alternating series). Suppose $0 < a_{n+1} \leq a_n$ for all n . Then $\sum (-1)^n a_n$ converges and its value S lies between any two partial sums, i.e., $S_{2n+1} \leq S \leq S_{2n}$.

Example. Give an example of a conditionally convergent series.

Theorem. e is irrational.

Define Rearrangement of a series.

Theorem. If $\sum a_n$ is absolutely convergent, then every rearrangement of this series converges (and to the same value).

Theorem. If $\sum a_n$ is conditionally convergent, then for every $x \in [-\infty, \infty]$ there is a rearrangement of the series which converges to x .

Theorem. If $a_n \geq 0$ for all n , then $\sum a_n$ converges iff there exists B such that every partial sum $S_n \leq B$.

Theorem. (Comparison Test) Suppose $0 \leq a_n \leq b_n$ for all n . Then if $\sum b_n$ converges, then $\sum a_n$ converges.

Theorem. (Limit Comparison Test) Suppose a_n and b_n are positive and $\lim \frac{a_n}{b_n} = c > 0$. Then $\sum a_n$ converges iff $\sum b_n$ converges.

Theorem. (Ratio Test) Suppose $a_n > 0$ for all n and

$$\lim \frac{a_{n+1}}{a_n} = r.$$

Then $\sum a_n$ converges if $r < 1$, and $\sum a_n$ diverges if $r > 1$.

Example. Give two examples of series with positive terms, such that $r = 1$ in the ratio test and one of the sequences converges and the other diverges.

Theorem. (Root Test) Suppose $a_n \geq 0$ for all n and

$$\limsup \sqrt[n]{a_n} = r.$$

Then $\sum a_n$ converges if $r < 1$, and $\sum a_n$ diverges if $r > 1$. In fact, if $r > 1$ there are infinitely many n with $a_n \geq 1$.

Example. Give two examples of series with positive terms, such that $r = 1$ in the root test and one of the sequences converges and the other diverges.

Theorem. (Integral Test) Suppose $f : [1, \infty) \rightarrow \mathbb{R}$ is a continuous function such that $\forall x, y \in [1, \infty)$ $x \leq y$ implies $f(x) \geq f(y) \geq 0$, i.e. f is monotone decreasing. Then $\sum f(n)$ converges iff $\int_1^\infty f(x)dx$ converges.

Corollary. $\sum \frac{1}{n^r}$ converges iff $r > 1$.

Theorem. (Radius of Convergence) For any power series $\sum a_n x^n$ there exists $R \in [0, \infty]$ such that $\sum a_n x^n$ converges absolutely for every x with $|x| < R$ and diverges for every x with $|x| > R$. Furthermore, the convergence is uniform on every interval of the form $[-r, r]$ with $r < R$.

Theorem. If $f_n : R \rightarrow \mathbb{R}$ are continuous (R is a rectangle in $\mathbb{R}^2$ or an interval in $\mathbb{R}$) and f_n converges to f uniformly on R , then

$$\iint_R f = \lim_{n \rightarrow \infty} \iint_R f_n.$$

Theorem. Suppose f_n converges uniformly to f on an open set $D \subseteq \mathbb{R}$ and each f_n is C' and f'_n converges uniformly to g on D . Then f is differentiable and $f'(x) = g(x)$ for every $x \in D$.

Corollary. If $f(x) = \sum a_n x^n$ has a nonzero radius of convergence, then the series $g(x) = \sum n a_n x^{n-1}$ has the same radius of convergence and $f' = g$. Similarly, If $f(x) = \sum a_n x^n$ has a nonzero radius of convergence, then the series $g(x) = \sum \frac{a_n}{n+1} x^{n+1}$ has the same radius of convergence and $g' = f$.

Corollary. If $f(x) = \sum a_n x^n$ has a nonzero radius of convergence, then f has infinitely many continuous derivatives (C^∞).

Corollary. If $f(x) = \sum a_n x^n$ has a nonzero radius of convergence, then $a_n = \frac{f^n(0)}{n!}$ for every n .

Corollary. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ for $|x| < 1$.

Corollary. $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| < 1$.

Theorem. (Abel) If $\sum a_n x^n$ has a nonzero radius of convergence R , and $\sum a_n x^n$ converges for $x = R$, then $\sum a_n x^n$ converges uniformly on the interval $[0, R]$.

Corollary. If $f(x) = \sum a_n x^n$ has a nonzero radius of convergence R , and $\sum a_n x^n$ converges for $x = R$, then f is continuous on $[0, R]$.

Corollary. $\ln(2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$.

Corollary. $\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$.