

Define f_n converges to f pointwise.

Define f_n converges to f uniformly.

Example. Give an example of a sequence of functions f_n which converges to a function f pointwise, but not uniformly.

Theorem. If $f_n : D \rightarrow \mathbb{R}^m$ converges uniformly to a $f : D \rightarrow \mathbb{R}^m$ and each f_n is continuous, then f is continuous.

Theorem. (Weierstrass) There exists a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is continuous, but not differentiable at any $x \in \mathbb{R}$.

Theorem. (Peano) There exists a continuous function $f : I \rightarrow I^2$ which maps $I = [0, 1]$ onto the unit square I^2 .

Define $\frac{\partial f}{\partial x}(p_0)$

Define $Df(p_0)$

Define f is C' .

Define f is differentiable (for a function mapping an open subset of $\mathbb{R}^d$ into $\mathbb{R}^m$). (p.345)

Lemma. Suppose $\frac{\partial f}{\partial x} \frac{\partial f}{\partial y}$ exist for every $p \in B(p_0, \delta)$. Then $\forall p \in B(p_0, \delta) \exists p_1, p_2 \in B(p_0, \delta)$ such that

$$f(p) - f(p_0) = \frac{\partial f}{\partial x}(p_1)\Delta x + \frac{\partial f}{\partial y}(p_2)\Delta y$$

Theorem. If f is C' , then f is differentiable.

Define the directional derivative $D_u f(p_0)$.

Theorem. If f is differentiable and u a unit vector, then

$$D_u f(p_0) = Df(p_0) \bullet u.$$

Define $\|A\|$ for A a $m \times d$ matrix.

Theorem. $\|Ap\| \leq \|A\| \|p\|$ for every $p \in \mathbb{R}^d$.

Theorem. If f is differentiable, then f is continuous (for a function mapping an open subset of $\mathbb{R}^d$ into $\mathbb{R}^m$).

Example. Give an example of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for every $p \in \mathbb{R}^2$ both $\frac{\partial f}{\partial x}(p)$ and $\frac{\partial f}{\partial y}(p)$ exist but f is not continuous at some point (hence it is not differentiable).

Example. Give an example of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for every $p \in \mathbb{R}^2$ both $\frac{\partial f}{\partial x}(p)$ and $\frac{\partial f}{\partial y}(p)$ exist, and at some point p neither partial derivative is continuous, but f is differentiable at p .

Theorem. Suppose $f : D \rightarrow \mathbb{R}^m$ where $D \subseteq \mathbb{R}^d$ is open, $p_0 \in D$, A is an $m \times d$ matrix, and R defined by:

$$f(p) - f(p_0) = A \bullet (p - p_0) + R$$

and R has the property that

$$\lim_{p \rightarrow p_0} \frac{\|R\|}{\|\Delta p\|} = 0$$

Then the partial derivatives of the component functions of f exist at p_0 and $Df(p_0) = A$.

Theorem. Chain rule. (p.136 and p.346)

Theorem. Suppose g is $C^{(n+1)}$ on the interval $(-r, r)$, $x \in (-r, r)$, and R is defined by:

$$g(x) = g(0) + g'(0)x + \frac{g^{(2)}(0)}{2}x^2 \cdots + \frac{g^{(n)}(0)}{n!}x^n + R.$$

Then for some ζ strictly between x and 0

$$R = \frac{g^{(n+1)}(\zeta)}{(n+1)!} x^{n+1}.$$

Theorem. If f is $C^{(n+1)}$ on some $B(p_0, r)$ for $p_0 \in \mathbb{R}^2$, $p \in B(p_0, r)$, and R is defined by:

$$f(p) = \sum_{k=0}^n \frac{1}{k!} (\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^k f(p_0) + R.$$

Then for some q on the line segment joining p to p_0

$$R = \frac{1}{(n+1)!} (\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^{n+1} f(q).$$

Theorem. Min-Max-Saddle point tests, see page 156-157.