

Keep this sheet. Use your own paper for the answers.

Results proved in class

1. Suppose $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b) . Suppose also that $g'(x) \neq 0$ for every x in (a, b) . Prove that there exists a $c \in (a, b)$ such that (10 points)

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

2. Prove that if $f_n : D \rightarrow \mathbb{R}^m$ converges uniformly to a $f : D \rightarrow \mathbb{R}^m$ and each f_n is continuous, then f is continuous. Begin by defining uniform convergence. (10 points)
3. Prove the following theorem. If f is C' , then f is differentiable. Begin by defining C' and differentiable. (10 points)
4. Prove the following. Suppose g is $C^{(n+1)}$ on the interval $(-r, r)$, $x \in (-r, r)$, and R is defined by: (10 points)

$$g(x) = g(0) + g'(0)x + \frac{g^{(2)}(0)}{2}x^2 \cdots + \frac{g^{(n)}(0)}{n!}x^n + R.$$

Then for some ζ strictly between x and 0

$$R = \frac{g^{(n+1)}(\zeta)}{(n+1)!}x^{n+1}.$$

Exercises

5. Prove that the derivative of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ has the intermediate value property, i.e., for any $a < b$ and v if v is between $f'(a)$ and $f'(b)$, then for some $c \in (a, b)$ we have $f'(c) = v$. (10 points)
6. Find and classify the critical points of $f(x, y) = x^2 + y^2 + xy^2$. (10 points)
7. (a) Give an example of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist for every point $p \in \mathbb{R}^2$, but f is not differentiable at $(0, 0)$. (10 points)
- (b) Give an example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is differentiable, but f' is not continuous.

Be sure and prove that your examples have the required properties.

New problems

8. Give an example of a sequence $f_n : \mathbb{R} \rightarrow \mathbb{R}$ of differentiable functions such that f_n converges uniformly to a function f which is not differentiable. (10 points)
9. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and its inverse function $f^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are differentiable, and $f(p) = q$. Prove that the matrix $Df^{-1}(q)$ is the inverse of the matrix of $Df(p)$. (10 points)
10. Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and for every (x, y) we have $|f(x, y)| \leq |xy|$. Prove that f is differentiable at $(0, 0)$. (10 points)