

Keep this sheet. Use your own paper for answers.

Results proved in class

1. Prove that $\forall x, y \in \mathbb{R}$ if $x < y$, then $\exists r \in \mathbb{Q}$ such that $x < r < y$. (7 points)
2. Define p is a cluster point of the sequence $\langle p_n : n \in \mathbb{N} \rangle$. (3 points)
3. Define $\langle p_n : n \in \mathbb{N} \rangle$ is a bounded sequence. (3 points)
4. Prove that every bounded sequence in $\mathbb{R}$ has a cluster point. (7 points)
5. Define f is continuous at p_0 where $f : D \rightarrow \mathbb{R}^m$ and $p_0 \in D \subseteq \mathbb{R}^d$. (3 points)
6. Define f is continuous where $f : D \rightarrow \mathbb{R}^m$ and $D \subseteq \mathbb{R}^d$. (3 points)
7. Define $f(D)$ where f is as above. (3 points)
8. Define $C \subseteq \mathbb{R}^n$ is a compact set. (3 points)
9. Prove that if $f : D \rightarrow \mathbb{R}^m$ is continuous and $D \subseteq \mathbb{R}^d$ is compact, then $f(D)$ is compact. (7 points)

Exercises

10. Prove that for every $p, q \in \mathbb{R}^d$ that $\|p - q\| \geq \|p\| - \|q\|$. (7 points)
11. Define $\langle p_n : n \in \mathbb{N} \rangle$ is a Cauchy sequence. (3 points)
12. Give an example of a sequence $\langle x_n : n \in \mathbb{N} \rangle$ of real numbers such that (7 points)

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad |x_{n+1} - x_n| < \epsilon$$

but $\langle x_n : n \in \mathbb{N} \rangle$ is not a Cauchy sequence. (Prove that your example is not Cauchy.)

13. Prove that if $C \subseteq \mathbb{R}$ is a nonempty closed and bounded set, then C has a maximum element, i.e., $\exists x \in C \quad \forall y \in C \quad y \leq x$. (7 points)

New problems

14. Define $U \subseteq \mathbb{R}^d$ is open in $\mathbb{R}^d$. (3 points)

15. Define $C \subseteq \mathbb{R}^d$ is closed in $\mathbb{R}^d$. (3 points)

16. Prove (a) iff (b) where (7 points)

(a) C is closed in $\mathbb{R}^d$

(b) $\forall p \in \mathbb{R}^d$ if $\forall \epsilon > 0 \exists q \in C$ with $\|p - q\| < \epsilon$, then $p \in C$.

17. Let $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ be any countable enumeration of the rational numbers. Define the function $f : \mathbb{R} \rightarrow \mathbb{R}$ by the rule: (7 points)

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/n & \text{if } x = r_n \end{cases}$$

Prove that f is discontinuous at each rational and continuous at each irrational.

18. Define f is uniformly continuous where $f : D \rightarrow \mathbb{R}^m$ and $D \subseteq \mathbb{R}^d$. (3 points)

19. Suppose that $f : D \rightarrow \mathbb{R}^m$ is uniformly continuous where $D \subseteq \mathbb{R}^d$ and $\langle p_n : n \in \mathbb{N} \rangle$ is a Cauchy sequence of elements of D . Prove that (7 points)

$$\langle f(p_n) : n \in \mathbb{N} \rangle$$

is a Cauchy sequence.

20. Suppose that $g : \mathbb{R} \rightarrow \mathbb{R}$ is a function and (7 points)

$$G = \{(x, y) \in \mathbb{R}^2 : y = g(x)\}$$

is its graph. Prove that if g is continuous, then G is closed.