

Keep this sheet. Show all your work and explain your answers.

1. Of 400 employees at a factory, all worked for three, four, or five days during a certain week (including Sunday). Show that at least 5 of the employees must have been present on exactly the same days. (10 points)
2. What is the largest power of ten which divides the number $30!$? (10 points)
3. What are the mobile integers in $\overleftarrow{6} \overleftarrow{5} \overrightarrow{3} \overleftarrow{1} \overrightarrow{2} \overleftarrow{4} \overrightarrow{7} ?$ (10 points)
What is the are the next three permutations after it (including the arrows)?
4. Let the set X be the set of all positive integers less than or equal to 1000 which are divisible by 3, 4, or 5. What is $|X|$? (10 points)
5. Let $h_{n+1} = h_n + 6h_{n-1}$ and $h_0 = 2$ and $h_1 = 1$. Find a formula for h_n . (10 points)
6. How many paths are there from $(0, 0, 0)$ to $(5, 5, 7)$ if each move is one of the following: (10 points)
 - $(x, y, z) \rightarrow (x + 1, y, z)$, or
 - $(x, y, z) \rightarrow (x, y + 1, z)$, or
 - $(x, y, z) \rightarrow (x, y, z + 1)$.

And at all times we have $x \leq y$?

7. (10 points)

$$\begin{aligned}A_1 &= \{a, b\} \\A_2 &= \{a, b, c, d\} \\A_3 &= \{d, e, f\} \\A_4 &= \{b, c\} \\A_5 &= \{a, c\} \\A_6 &= \{a, b, c\}\end{aligned}$$

Does the sequence $\langle A_1, A_2, A_3, A_4, A_5, A_6 \rangle$ have a system of distinct representatives (SDR)? If so, what is it? If not, show why not.

8. Let (V, E) be a bipartite graph. Hence V is the disjoint union of two sets A and B so that no two vertices in A are adjacent and no two vertices in B are adjacent. Suppose p is a positive integer such that every vertex of A has degree $\geq p$ and every vertex of B has degree $\leq p$. Prove that $|A| \leq |B|$. (10 points)

9. Suppose $G = (V, E)$ is a graph such that $|V| = n$ and (10 points)

$$|E| > \binom{n-1}{2}.$$

Show that for any two distinct vertices $u, v \in V$, either u and v are adjacent or there exists a vertex $w \in V$ such that w is adjacent to both u and v .

10. Do **one** of the following: (10 points)

1. Prove Euler's formula: $|V| + |F| - |E| = 2$.
2. Prove that the graph K_5 is not planar.
3. Prove that the graph $K_{3,3}$ is not planar.

If you do (2) or (3) you may assume (1) without proof.