

1. Prove that for any integer  $n > 1$ :

(15  
points)

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} + \cdots + (-1)^{n-1}n\binom{n}{n} = 0.$$

2. Construct a partition of all subsets of  $\{1, 2, 3\}$  into symmetric chains.

(10  
points)

3. What is the coefficient of  $x^2y^3z^4$  in  $(6x + 5y + 4z)^9$ ?

(10  
points)

4. In how many ways can 6 nonattacking rooks be placed on the  $6 \times 6$  chess board below with forbidden positions shown by X?

(15  
points)

|   |   |   |  |  |   |
|---|---|---|--|--|---|
| X |   |   |  |  |   |
| X |   |   |  |  | X |
|   |   |   |  |  |   |
|   | X | X |  |  |   |
|   | X | X |  |  |   |
|   |   |   |  |  |   |

5. Given four pairs of gloves, in how many ways can they be distributed to four persons so that each person receives a left glove and a right glove from a different pair, i.e., a nonmatching pair. (15 points)

6. Let  $h_n$  denote the number of “comparisons” required to sort a list of  $2^n$  numbers using the “merge sort” technique. Note that  $h_0 = 0$ . A merge sort is performed recursively by partitioning the list of  $2^{n+1}$  into two sublists of  $2^n$  each, sorting each of these sublists, and then merging the two lists. Assume that it takes  $2^{n+1}$  comparisons to merge the two sorted sublists into a single list. Write a recurrence relation for  $h_{n+1}$  and solve it. (15 points)

7. Let  $h_{n+1} = 3h_n - 2h_{n-1}$  and  $h_0 = 4$  and  $h_1 = 5$ . Find a formula for  $h_n$ . (10 points)

8. Let  $h_n$  be the number of solutions in nonnegative integers of the equation  $a + b + c = n$  where  $a$  is even,  $b$  is odd, and  $c$  is less than three. Find the generating function of this sequence and express it as a rational function, i.e., the quotient of polynomials. (10 points)