Instructions

No notes, no books, no calculators, no cell phones, no pagers, no electronic devices of any kind.

In problems 2 and 3 you do not need to simplify your answer.

Name____________________________________

1. Given the inversion sequence $(2, 4, 2, 1, 1, 0)$, construct the permutation of $\{1, 2, 3, 4, 5, 6\}$ which it corresponds to.

2. There are 20 balls to be put into 6 numbered buckets.

(1) How many are possible if the balls are distinguishable, i.e., the balls are numbered 1 thru 20?

(2) How many ways are possible if the balls are indistinguishable?

(3) How many ways are possible if the balls are indistinguishable and each bucket must get at least two balls?

3. (a) How many ways are there to divide 12 people into 3 groups, one each of size 3, 4, and 5?

(b) Suppose two of the people, Alice and Barbara, want to be in different groups. Then how many ways can it be done?

4. Let $n > 1$ be an integer. Prove that given $n + 1$ integers there must be two of them whose difference is divisible by $n$.

5. Let $W$ be the set of all words of length three which contain only the letters $a$, $b$, and $c$. Define the binary relation $E$ on $W$ by

$w_1 \; E \; w_2$ iff $w_1$ and $w_2$ contain the same number of $a$'s, same number of $b$'s, and the same number of $c$'s.

(a) Prove that $E$ is an equivalence relation.

(b) What is the equivalence class containing the word abc ?

(c) How many equivalence classes are there?

Hint: List a representative of each class.

Answers

1. 641352

2. (1) $6^{20}$ (2) $\binom{25}{5}$ (3) $\binom{13}{5}$

3. (a) $\binom{12}{3,4,5}$ (b) $\binom{12}{3,4,5} - \left[\binom{10}{1,4,5} + \binom{10}{3,2,5} + \binom{10}{3,4,3}\right]$

4. Recall that $r = a \bmod n$ is the remainder after dividing the integer $a$ by $n$. Equivalently,

$$r = a \bmod n \qquad \text{iff} \qquad 0 \leq r < n \text{ and } \exists q \in \mathbb{Z} \;\; a = qn + r$$

Now suppose we are given a set $A$ of integers with $|A| = n + 1$. Since there are only $n$ possible remainders $0, 1, \ldots, n - 1$ by the Pidgeon Hole Principle there must be distinct integers $a, b \in A$ such that

$$a \bmod n = b \bmod n = r.$$

But then $a = qn + r$ and $b = \hat{q}n + r$ for some integers $q$ and $\hat{q}$ and so

$$a - b = (q - \hat{q})\, n$$

is divisible by $n$.

5. (1) State the definitions of reflexive, symmetric, and transitive and note the $E$ has each of these properties.
(2) abc acb bac bca cab cba
(3) There are 10. A list of representatives would be
abc    aab aac    bba bbc    cca ccb    aaa bbb ccc