(1) a) you choose 6 of your 20 elements to be in the subset, so there are $\binom{20}{6}$ ways to make a subset of size 6. That is the same as $\boxed{\dfrac{20!}{6!\,14!}}$

for b, c, assume $x_i < x_j$ if $i < j$

b) There is a one to one correspondance between $X = \{\{x_1, x_2, x_3, x_4, x_5, x_6\} : |x_i - x_j| \leq 1 \text{ iff } x_i = x_j,\ x_k \in \{1, 2, \dots 20\} \text{ for all } k\}$ and $Y = \{\{y_1, y_2, y_3, y_4, y_5, y_6\} \subseteq \{1, 2, \dots 15\}\}$. Call it f.

Then $f : \{y_1, y_2, \dots, y_6\} \longrightarrow \{y_1, y_2+1, y_3+2, \dots y_6+5\} = \{x_1, x_2, \dots, x_6\}$

so $|X| = |Y|$. So there are $\boxed{\binom{15}{6} = \dfrac{15!}{6!\,9!}}$ ways to do this.

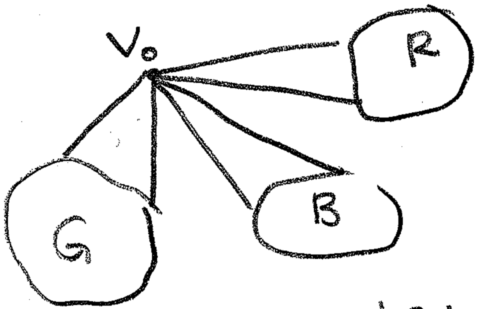
c) There is a 1-1 correspondance between $X = \{\{x_1, x_2, \dots x_6\} \subseteq \{1, 2, \dots 20\} : |x_i - x_j| \leq 2 \Rightarrow x_i = x_j\}$ and $Y = \{\{y_1, y_2, \dots y_6\} \subseteq \{1, 2, \dots 10\}\}$ call the map g. Then

$g : \{y_1, y_2, \dots y_6\} \longrightarrow \{y_1, y_2+2, y_3+4, y_4+6, y_5+8, y_6+10\} = \{x_1, x_2, \dots, x_6\}$

Then $|X| = |Y| = \boxed{\binom{10}{6} = \dfrac{10!}{6!\,4!}}$

2

If you have a K_{17} colored red, blue and green, fix vertice V_0



Let R be all vertices connected to V_0 with a red edge, B be all vertices connected to V_0 with a blue edge and G be all vertices connected to V_0 with a green edge. Then you know $|R| \geq 6$ or $|B| \geq 6$ or $|G| \geq 6$, otherwise you would have $|R| \leq 5$, $|B| \leq 5$ and $|G| \leq 5$, Hence $|R| + |B| + |G| \leq 15$ & if you add back in V_0, you have atmost 16 vertices which is impossible. So you must have $|R| \geq 6$, $|B| \geq 6$ or $|G| \geq 6$.

<u>Case 1</u>:
$|R| \geq 6$.

<u>Case 1a</u>:
There are $X, Y \in R$ where the edge $\{X, Y\}$ is colored red. Then you have a red K_3 made by connecting V_0, X & Y.

<u>Case 1b</u>:
There are <u>no</u> edges in R colored red. Then you have atleast a K_6 with only 2 colors. We know $K(3,3) = 6$, so there mustbe a blue K_3 or a green K_3.

<u>Case 2</u> $|B| \geq 6$ and case 3: $|G| \geq 6$ are Symmetric.

$\square$

3 (3)

Let $A_1, A_2, \dots A_n$ be a partition on a set X. Then define R on $X \times X$ by xRy iff $\exists k$ $1 \leq k \leq n$ s.t. $x \in A_k$ and $y \in A_k$. Claim: R is an equivalence relation.

Pf:
A relation $S \subseteq A \times A$ is an equivalence relation iff it is reflexive, symmetric and transitive. Reflexive means for every $a \in A$ aSa. Symmetric means for every $a, b \in A$, if aSb, then bSa. Finally transitive means for every $a, b, c \in A$, if aSb and bSc, then aSc.
Our relation R satisfies these criteria:

Reflexive:
take $x \in X$. Since $A_1, A_2 \dots A_n$ is a partition, meaning $A_1 \cup A_2 \cup \dots \cup A_n = X$ and $A_i \neq \emptyset$ for all $i = 1, 2, \dots n$ and $A_i \cap A_j \neq \emptyset \Rightarrow i = j$, $x \in A_i$ for some i. So xRx because x is in a set of the partition & x is in the same part of the partition as itself. So R is reflexive.

Symmetric:
Take $x, y \in X$ s.t. xRy. Then $\exists k \in \{1, 2, \dots n\}$ s.t. $x \in A_k$ and $y \in A_k$. That is the same as saying $y \in A_k$ and $x \in A_k$. So yRx. Hence R is symmetric.

Transitive:
Take $x, y, z \in X$ s.t. xRy and yRz. That means $\exists k$ $1 \leq k \leq n$ s.t. $x \in A_k$ and $y \in A_k$. Also, $\exists j$, $1 \leq j \leq n$ s.t. $y \in A_j$ and $z \in A_j$. Since y is an element of both A_k & A_j $y \in A_k \cap A_j$. So $A_k \cap A_j \neq \emptyset$, but since $A_1, A_2 \dots A_n$ is a partition, That means $A_k = A_j$. So that means $z \in A_k$. Which implies xRz. So R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation. $\square$

 [redacted]

an antichain, A, contains elements where no two elements are a subset of eachother

4) case 1: Let x be $\in A$ and $|x| = \emptyset$, since $\emptyset$ is a subset of every element $\{1,2,3,4\}$, A cannot contain any other set $\therefore |A| = 1$

case 2: Let x be $\in A$ and $|x| = 4$, since x contains the entire set $\{1,2,3,4\}$ no other set can be added to A w/out being a subset of x $\therefore |A| = 1$

case 3: Let x be $\in A$ and $|x| = 1$ since x contains one element, i any additional sets to add to x must be from $\{1,2,3,4\}/\{i\}$ which $|\{1,2,3,4\}/\{i\}| = 3$ the max antichain from a set of 3 is $\binom{3}{\lfloor 3/2 \rfloor} = 3$, therefore $|A| \leq 1+3 = 4$

case 4: Let x be $\in A$ and $|x| = 3$. If A contains more than one set of 3 $|A| \leq \binom{4}{3} = 4$ ie $\{(1,2,3), (124), (134), (234)\}$

If y is also in A and $|y| = 1$ we defer to case 3 which is an A w/ an element of size 1 and that $|A| \leq 4$

If A is made up of $|x| = 3$ and sets of 2 (ie $\{1,4\}$) then if $x \equiv \{i,j,k\}$ the sets of two can only be made from $\{1,2,3,4\}$ but cant include ij, ik, jk $\therefore$ the # of two sets that can be added to A $= \binom{4}{2} - 3 = 6 - 3 = 3$ $\therefore |A| \leq 1+3 = 4$

$\therefore$ if $x \in A$ and $|x| = 3$, $|A| \leq 4$

case 5 The number of 2-element subsets from $\{1,2,3,4\}$ that can fill an antichain is $\binom{4}{2} = 6$

ex: $\{(12), (13), (14), (23), (24), (34)\}$

If the A contains any other size set it falls in cases 1-4 and $A < 6$ $\therefore$ the only A of size 6 is the partial order of all 2-element subsets.

5. There are $n!$ ways to arrange n numbers.
Partition the arrangements according to how many numbers are in their natural position; that is, how many numbers match the index of the spot they're in. For example, in the permutation $\underline{1}\ \underline{3}\ \underline{2}$, 1 is in the 1^{st} position and is therefore in its natural position, while 2 and 3 are not.
For the partition where k elements are in their natural position:
choose the k elements: $\binom{n}{k}$
and the rest must not be in their natural position, and so form a derangement of size $n-k$: D_{n-k}
for total $\binom{n}{k} D_{n-k}$.
Sum over all the possible partitions to get
$$n! = \sum_{k=0} \binom{n}{k} D_{n-k}.$$

(77) Determining an onto function from an n-set to a k-set requires two steps:

<u>step 1</u>: Partition the n-set into k non-empty parts to determine which elements in the n-set are mapped to the same element in the k-set.

$\Rightarrow$ # of ways to do this is $S(n,k)$ (by definition of $S(n,k)$)

<u>step 2</u>: To determine which part of the k-partition is mapped to which element in the k-set, this is ~~basically~~ equivalent to permutations of the k parts from the partition in step 1.

$\Rightarrow$ # of ways to do this is $k!$

Thus, number of ~~onto~~ onto functions from an n-set to a k-set is $k!\,S(n,k)$.

6

So $f_0=0$ and $f_1=1$ and $f_{n+1}=f_n+f_{n-1}$ for all $n\in\mathbb{N}_1$

Then, $f_0\equiv 0 \mod 4$ and $f_1\equiv 1 \mod 4$. When $4|x$ $x\equiv 0 \mod 4$. Since any number is equivalent to the remainder upon division by n in mod n. Therefore you can add f_n+f_{n-1} in mod 4 and get f_{n+1} in mod 4. In other words

n	f_n mod 4
0	0
1	1
2	1
3	2
4	3
5	1
6	0
7	1
8	1

$f_{n+1} \mod 4 = (f_n+f_{n-1}) \mod 4$

You notice 0 followed by 1 is repeated at 6, & since $f_{n+1}=f_n+f_{n-1}$, you now the pattern from $n=0$ to $n=5$ will repeat. That means $f_n\equiv 0 \mod 4$ iff n is divisible by 6. That means $4|f_n$ iff $6|n$. □

8

Let $\mathcal{A}=\{A_1,A_2,\dots A_n\}$ be a family of sets with an SDR. Let $x\in A_1$. Then there exists an SDR

$a_1\ a_2\ a_3 \dots a_n$ where $a_i\in A_i$ for all $i=1,2,\dots n$

and $a_i=a_k \Rightarrow i=k$. Then $\exists$ an SDR containing x.

Case 1:

$x=a_i$ for some i. Then you are done.

Case 2:

x is not an element of the SDR $a_1 a_2 \dots a_n$. Then replace a_1 with x. Since $x\in A_1$ & $x\neq a_i$ for $i=1,2,\dots n$, $x\ a_2\ a_3 \dots a_n$ is still an SDR.

However, it may not be possible for x to represent A_1. Here is a counter example.

$A_1=\{x,1,3\}$ $A_2=\{2,4\}$, $A_3=\{x\}$ $A_4=\{3\}$. The only SDRs are $1,2,x,3$ and $1,4,x,3$.
Then x must represent A_3, so it can't represent A_1. □