A.Miller
M475
Fall 2010

Homework problems are due in class one week from the day assigned (which is in parentheses). Please do not hand in the problems early.

1. (1-20 W) A book shelf holds 5 different English books, 4 different French books, and 8 different German books.

(a) How many ways are there to choose 3 books, one from each language?

(b) How many ways are there to choose 2 books, each from a different language?

2. (1-22 F) You have eight different types of candy, for example, chocolate bars, lolly pops, gum drops, and so forth for 5 more.

(a) How many ways are there to give Tom, Dick, and Harry each a piece of candy assuming each gets a different type. For example, you could give a lolly pop to Tom, chocolate bar to Dick, and cinnamon roll swirl to Harry.

(b) How many ways are there to give Tom, Dick, and Harry each a piece of candy but not necessarily different types. For example, you could give a lolly pop to both Tom and Harry, and chocolate bar to Dick.

3. (1-22 F) Suppose A and B are n-sets.

(a) How many functions are there from A to B which are not one-to-one?

(b) How many functions are there from A to B which are onto?

4. (1-25 M) How many permutations are there of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ such that 1 is next to 2, 3 is next to 4, 5 is next to 6, and 7 is next to 8?

For example, $\langle 8, 7, 1, 2, 4, 3, 5, 6\rangle$.

5. (1-25 M) How many ways can 4 women and 4 men be seated around a circular table so that no two women are seated next to each other.

6. (1-25 M) How many 5 card poker hands contain 2 pairs?

Warning: do not count 4 of kind or 3 of kind or full-houses.

7. (1-27 W) How many ways can Tom, Dick, Harry, and Bob be seated in a row of 15 numbered seats so that between any two occupied sets there are at least two unoccupied seats.

8. (1-27 W) How many ways can 6 women and 3 men seat themselves around a circular table so that no two men are sitting next to each other? Assume that the three men are indistinguishable, i.e., MAN, MAN, MAN, but the six women are distinguishable, i.e., Alice, Mary, Sally, Sue, Dolly, and Lee.

9. (1-29 F) Suppose A is a set with five elements.
(a) How many functions are there $f : A \to \{0, 1, 2, \ldots\}$ such that

$$\sum_{a \in A} f(a) = 10?$$

(b) How many functions are there $f : A \to \{1, 2, 3, \ldots\}$ such that

$$\sum_{a \in A} f(a) = 10?$$

10. (1-29 F) Suppose that $k + 1 \leq n$. Prove that

$$\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k+1}{k} + \binom{k}{k}.$$

11. (2-1 M) Suppose a seven letter word is made up of the letters from $\{a, b, c\}$ is chosen at random. What is the probability that it will contain exactly three a's? For example, it could be: bacaabb.

12. (2-3 W) Please email me (miller@math.wisc.edu) a digital photo of yourself (jpg). If there is someone in the picture besides you, be sure and mention in the email which one is you.

13. (2-3 W) Suppose $A \subseteq \{1, 2, 3, \ldots, 12\}$ and $|A| \geq 9$.
(a) Prove there must be an integer x such that $x, x+1, x+2$ are all in A.
(b) Give an example of an A with $|A| = 8$ for which there is no such x.

14. (2-5 F) The hours on a clock face $\{1, 2, \ldots, 12\}$ are randomly permuted. Prove that there must be 3 hours in a row whose sum is at least 20.

15. (2-8 M) Prove that if the edges of the complete graph on 17 vertices, K_{17}, are colored with three colors, then there are three vertices, all of whose edges are the same color.

16. (2-8 M) Find a coloring of the edges of K_{10} with three colors which has no monochromatic triangle.

17. (2-10 W) Find the permutation which occurs next in lexicographical order after:

$$(3, 8, 1, 6, 5, 9, 7, 4, 2)$$

18. (2-10 W) Find the inversion sequence of the permutation:

$$(4, 7, 1, 9, 6, 2, 5, 3, 8)$$

19. (2-10 W) Find the permutation which has the inversion sequence:

$$(6, 7, 2, 5, 3, 1, 1, 1, 0)$$

20. (2-12 F) Let $A_1, A_2, \ldots, A_N$ list all subset of $\{1, 2, \ldots, n\}$ using the increment and carry listing where $N = 2^n$. Let $f : \{1, \ldots, N\} \to \{1, \ldots, N\}$ be the function defined by

$$A_{f(k)} = \overline{A}_k$$

Determine the function f and verify that your function is correct for $n = 4$. If you prefer may list the sets $A_0, A_1, A_2, \ldots, A_N$ with $N = 2^n - 1$ and/or you may increment the right-most or the left-most digit with carry.

21. (2-12 F) Let I_n be the set of inversion sequences for the group of permutations S_n of $\{1, 2, \ldots, n\}$. Show that there is a listing of I_n:

$$\vec{i}_1, \vec{i}_2, \ldots, \vec{i}_N$$

where $N = n!$ such that for any $k < N$ the n-sequences $\vec{i}_k$ and $\vec{i}_{k+1}$ differ only on one coordinate, say l, and on that coordinate l they differ by exactly one, i.e.,

$$|i_{k,l} - i_{k+1,l}| = 1$$

Write down your list for $n = 4$ and next to each entry the permutation in S_4 with that inversion sequence. What property does the list of S_4 have?

22. (2-15 M)

1. How many binary relations R are there on the set $\{1, 2, \ldots, n\}$ which are reflexive?
2. How many binary relations R are there on the set $\{1, 2, \ldots, n\}$ which are irreflexive?
3. How many binary relations R are there on the set $\{1, 2, \ldots, n\}$ which are symmetric?
4. How many binary relations R are there on the set $\{1, 2, \ldots, n\}$ which are symmetric and irreflexive?

23. (2-15 M) Draw a Hasse diagram for the partial order:

$$(\{1, 2, 3, \ldots, 32\}, \trianglelefteq)$$

where $x \trianglelefteq y$ iff x divides y.

24. (2-17 W) Prove:

$$\frac{2^{n+1} - 1}{n+1} = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}$$

25. (2-19 F) Prove:

$$\binom{3n}{n} = \sum_{a+b+c=n} \binom{n}{a} \binom{n}{b} \binom{n}{c}$$

26. (3-5 F) Prove: $K_8 \to (C_4, C_4)$. This means that if the edges of K_8 are colored red or blue, then there are four distinct vertices v_1, v_2, v_3, v_4 such that the edges $\{v_1, v_2\}$,$\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_4, v_1\}$ are either all red or all blue. (C_4 is the 4 cycle graph.)

27. (3-10 W) How many ways can you put 20 indistinguishable balls into 4 distinguishable buckets so that every bucket has at least one ball and no bucket has more than 7 balls?

28. (3-10 W) How many ways can 5 red balls and 5 blue balls be put into 3 distinguishable boxes so that no box is empty?

29. (3-12 F) 6 men are randomly returned their 6 hats. Find the following probabilities:

(a) No man gets his own hat.

(b) Every man gets his own hat.

(c) Exactly one man gets his own hat.

(d) At least two men get their own hat.

For the above give the exact numerical answer. Repeat (a) thru (d) for any $n \geq 6$ (n men with n hats) and express your answers in terms of D_n.

30. (3-12 F) How many permutations of 1,2,3,4,5,6,7 are there which do not contain any block of the form 123, 345, or 567? For example, 4123576 would be bad because of the 123, but 3217654 would be good.

31. (3-15 M) Let $\mathcal{G}_n$ be the set of all $X \subseteq \{1, 2, \ldots, n\}$ which do not contain a successive pair of integers. Put $g_n = |\mathcal{G}_n|$. Prove that the g_n satisfy the recurrence relation:

$$g_{n+1} = g_n + g_{n-1}$$

By determining the first two values of the g_n show that $g_n = F_{n+2}$ where F_n is the Fibonacci sequence.

32. (3-15 M) Let

$$g_{n,k} = |\{X \in \mathcal{G}_n : |X| = k\}|$$

Prove that

$$g_{n,k} = \binom{n-k+1}{k}$$

and conclude that the Fibonacci sequence satisfies:

$$F_{n+2} = \sum_{k=0}^{n} \binom{n-k+1}{k}$$

where $\binom{a}{b} = 0$ whenever $b > a$.

33. (3-17 W) Given the recurrence relation

$$a_{n+1} = a_n + 6a_{n-1}$$

(a) Find r, s such that $a_n = \alpha r^n + \beta s^n$ is the general solution.

(b) Given $a_0 = 1$ and $a_1 = 2$ find α and β.
(c) Given $a_0 = 2$ and $a_1 = 1$ find α and β.

34. (3-17 W) Given the recurrence relation $a_{n+1} = a_n + 6a_{n-1}$ let

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

be the associated generating function. Find polynomials p and q with

$$g(x) = \frac{p(x)}{q(x)}$$

35. (3-17 W) Given

$$\frac{x+1}{x^2+2x-3} = a_0 + a_1 x + a_2 x^2 + \cdots$$

find constants p and q such that for every $n = 1, 2, \ldots$

$$a_{n+1} = p\, a_n + q\, a_{n-1}$$

36. (3-22 M) Let

$$a_n = |\{(i,j) \ : \ i + 2j = n, \ i,j \geq 0\}|$$

and

$$g(x) = \sum_{n=0}^{\infty} a_n x^n.$$

(a) Show that $g(x)$ is a rational function of x, i.e., the ratio of polynomials.

(b) Using the method of partial fractions and the formula for geometric series, find an explicit formula for a_n.

37. (3-24 W) For $m \geq 2$ prove that

$$\sum_{k=1}^{m} \binom{m}{k} k^{m-1} (-1)^k = 0$$

38. (4-5 M) Solve the recurrence relation: $a_{n+2} = a_{n+1} + 6a_n + n^2$ with initial values: $a_0 = 0$ and $a_1 = 1$.

39. (4-5 M) Solve the recurrence relation: $a_{n+2} = a_{n+1} + 6a_n + 3^n$ with initial values: $a_0 = 0$ and $a_1 = 1$.

40. (4-7 W) How many paths are there from $(0, 0)$ to $(6, 6)$ such that each step consists of either moving from $(i, j) \mapsto (i+1, j)$ or $(i, j) \mapsto (i, j+1)$ and

(a) the path is on or below the diagonal.

(b) the path is on or above the diagonal.

(c) the path contains at least one point strictly below the diagonal and at least one point strictly above the diagonal.

(d) the path is strictly above the diagonal except at the points $(0, 0)$ and $(6, 6)$.

41. (4-9 F) Let C_n be the Catalan numbers. Give a combinatorial proof in terms of paths that

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0$$

Hint: Consider the penultimate place where the path visits the diagonal.

42. (4-14 W) For $k \geq 1$ define

$$g_k(x) = \sum_{n=k}^{\infty} S(n, k)x^n$$

Prove for $k \geq 2$ that

$$g_k(x) = x(g_{k-1}(x) + kg_k(x))$$

43. (4-14 W) Prove that

$$g_k(x) = \frac{x^k}{(1-x)(1-2x)(1-3x)\cdots(1-kx)}$$

44. (4-16 F) Give a combinatorial proof that Bell numbers satisfy:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

45. (4-19 M) Let

$$f(x) = e^{e^x - 1}$$

Prove that

$$f^{(n+1)}(x) = e^x \left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) \right)$$

46. (4-19 M) Prove that

$$e^{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

47. (4-21 W) Find

$$\sum_{1 \leq a_1 < a_2 < \ldots < a_{30} \leq 100} a_1 a_2 \cdots a_{30}$$

Express your answer in terms of a Stirling number.

48. (4-21 W) Calculate the partition numbers

(a) $p(8)$

(b) $p(16, 8)$

(c) $p(50, 42)$

49. (4-23 F) Prove that the number of partitions of the integer n such that each part is not divisible by three is the same as the number of partitions of n such that each part is repeated at most twice.

50. (4-28 W) Suppose $R \subseteq A \times B$ are finite sets and there exists $k \geq 1$ such that for every $a \in A$ and $b \in B$

$$|R^b| \leq k \leq |R_a|$$

Prove that R satisfies the Hall condition and hence there is a matching $M \subseteq R$. $R_a = \{y \in B : (a, y) \in R\}$ and $R^b = \{x \in A : (x, b) \in R\}$.