Alex Fu

4/21/2010 W

47. Express $\sum_{1 \le a_1 < a_2 < \dots < a_{30} \le 100} a_1 a_2 \dots a_{30}$ in terms of Stirling numbers. 47 on

48. Calculate 48 on

(a) $\rho(8)$

(b) $\rho(16,8)$

(c) $\rho(50,42)$

48 (a) $\rho(8) = \sum_{k=1}^{8} \rho(8,k)$

$= \rho(8,1) + \rho(8,2) + \rho(8,3) + \dots + \rho(8,8)$

$= 1 + 4 + \rho(8,3) + \rho(8,4) + \rho(8,5) + \rho(8,6) + 1 + 1$

$= 7 + \sum_{j=1}^{3}(5,j) + \sum_{j=1}^{4}(4,j) + \rho(7,4) + \rho(7,5)$

$= 7 + [1 + 2 + \rho(5,3)] + (1+2+1+1) + \rho(6,3) + \rho(6,4)$

$= 15 + 2 + \rho(6,3) + \rho(5,3)$

$= 17 + 2 + \sum_{j=1}^{3} \rho(3,j)$

$= 19 + 3$

$= 22$

(b) $\rho(16,8) = \sum_{j=1}^{8} \rho(8,j)$

$= \rho(8)$

$= 22$

(c) $\rho(50,42) = \sum_{j=1}^{42} \rho(8,j)$

$= \sum_{j=1}^{8} \rho(8,j)$

$= \rho(8)$

$= 22$

47. $\frac{[x]^{101}}{x} = x(x+1)(x+2)\dots(x+99)(x+100) \div x = (x+1)(x+2)\dots(x+100) = LHS$

$= \sum_{k=1}^{101} s(101,k)x^k \div x$

$= \sum_{k=0}^{100} s(101,k+1)x^k = RHS$

By comparing coefficient of x^{70} of both sides,

LHS : coefficient $= \sum_{1 \le a_1 < a_2 < \dots < a_{30} \le 100} a_1 a_2 a_3 \dots a_{100}$

RHS : coefficient $= s(101,71)$

$\therefore \sum_{1 \le a_1 < a_2 < \dots < a_{30} \le 100} a_1 a_2 \dots a_{30} = s(101,71)$

Lingfei Zhao.

OK

$\boxed{\text{HW 49}}$ Consider a partition of n into parts not divisible by 3.
We can write this as.

$$n = m_1 a_1 + m_2 a_2 + \cdots + m_j a_j$$

where $a_1, a_2, \cdots a_j$ are distinct integers, none divisible by 3.
For each m_i, it can be represented in base 3 as.

$$\left[\begin{array}{l} m_i = X_{i0} 3^0 + X_{i1} 3^1 + \cdots + X_{in} 3^n \\ \quad \text{where } 0 \le i(j) \le 2 \end{array} \right.$$

Therefore: $m_i a_i = X_{i0}[3^0 a_i] + X_{i1}[3^1 a_i] + \cdots X_{in}[3^n a_i]$
Because $3^\alpha a_i = 3^\beta a_j$ iff $\alpha = \beta$ and $i = j$, n can be partitioned into parts of size $[3^\alpha a_i]$ with repetition $X_{i\alpha}$. So we have a map from partions of n where no part is divisible by 3 to partitions of n where no part is repeated more than 2 times.
The map is also an injection. Let P_1 and P_2 be two distinct partitions of n where no part is divisible by 3. If there is a part size a in P_1 not in P_2, then P_1 gets mapped to a partition that contains a part of size $3^\alpha a$, while the image of P_2 contains no part of this size. If all the part sizes of P_1 and P_2 are the same, then some multiplicities must differ: for instance a is repeated m_1 times in P_1 and m_2 times in P_2. Since $m_1 \neq m_2$ they have distinct representations base 3, and it implies in the images of P_1 and P_2 some part size $[3^\alpha a]$ must be repeated a different number of times.

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ok Math 475 HW
07/07/2020.

50) Suppose $R \subseteq A \times B$ are finite sets and there exists $k \geq 1$ such that for every $a \in A$ and $b \in B$ $|R^b| \leq k \leq |R_a|$. Prove that R satisfies the Hall condition and hence there is a matching $M \subseteq R$.

$R_a = \{y \in B : (a, y) \in R\}$

$R^b = \{x \in A : (x, b) \in R\}$.

~~Pf Let i counts the number of girls, $1 \leq i \leq |X|$~~

~~(that is $a_i \in A$, $b' \in B$, $(a_i, b') \in R$)~~

~~and j counts the number of boys, $1 \leq j \leq |R(X)|$~~

~~(that is $b_j \in B$, $a' \in A$, $(a', b_j) \in R$)~~

Pf Count number of arrows from A to B,

$$\text{total} = \sum_{i=1}^{|X|} |R_{a_i}| = |R_{a_1}| + |R_{a_2}| + \dots + |R_{a_{|X|}}|$$

$$\geq \underbrace{k + k + \dots + k}_{|X|\text{-many}} \quad \left(\text{Since } |R_a| \geq k \ \forall a \in A.\right)$$

$$= |X| \cdot k \qquad (1)$$

Count number of arrows from B to A,

$$\text{total} \leq \sum_{j=1}^{|R(X)|} |R^{b_j}| = |R^{b_1}| + |R^{b_2}| + \dots + |R^{b_{R|X|}}|$$

some arrow may not reach to X

$$\leq \underbrace{k + k + \dots + k}_{|R(X)|\text{-many}} \quad \left(\text{Since } |R_b| \leq k \ \forall b \in B\right)$$

$$= |R(X)| \cdot k \qquad (2)$$

Since we are summing up, which are all the possible arrows,

$$\sum_{i=1}^{|X|} |R_{a_i}| \leq \sum_{j=1}^{|R(X)|} |R^{b_j}|$$

From (1) and (2),

$$|X| \cdot k \leq \sum_{i=1}^{|X|} |R_{a_i}| \leq \sum_{j=1}^{|R(X)|} |R^{b_j}| \leq |R(X)| \cdot k.$$

We thus have

$$|X| \cdot k \leq |R(X)| \cdot k \qquad (3)$$

Since $k \geq 1 > 0$, we can divide inequality (3), throughout by k, and get

$$|X| \leq |R(X)|$$

that is $|R(X)| \geq |X|$

This is satisfaction of the Hall Condition. ▨

By theorem (P. Hall, König), there is a matching $M \subseteq R$. ▨