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4/19/2010 M

45. Let $f(x) = e^{e^x - 1}$. Prove that $f^{(n+1)}(x) = e^x\left(\sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x)\right)$

46. Prove $e^{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$

45 ok
46 ok

45. $f^{(0+1)}(x) = \frac{d}{dx} e^{e^x-1}$
$= e^x \cdot e^{e^x-1}$
$= e^x f^{(0)}(x)$
$= e^x \sum_{k=0}^{0}\binom{0}{k} f^{(0)}(x)$

$f^{(1+1)}(x) = \frac{d}{dx} e^x \cdot e^{e^x-1}$
$= e^x e^{e^x-1} + e^x(e^x)(e^{e^x-1})$
$= e^x f^{(0)}(x) + e^x f^{(1)}(x)$
$= e^x \sum_{k=0}^{1}\binom{1}{k} f^{(k)}(x)$

$\therefore f^{(n+1)}(x) = e^x \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x)$ for $n = 0, 1$

Assume it is also true for $n = 0, 1, 2, \ldots, n$

For $n = k+1$, $f^{(n+1+1)}(x) = \frac{d}{dx} f^{(n+1)}(x)$
$= \frac{d}{dx} e^x \cdot \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x)$
$= e^x \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x) + e^x \cdot \sum_{k=0}^{n}\binom{n}{k} f^{(k+1)}(x)$
$= e^x\left[\binom{n}{0} f^{(0)}(x) + \sum_{k=1}^{n}\binom{n}{k} f^{(k)}(x) + \sum_{k=1}^{n}\binom{n}{k-1} f^{(k)}(x) + \binom{n}{n+1-1} f^{(n+1)}(x)\right]$
$= e^x\left[\binom{n+1}{0} f^{(0)}(x) + \sum_{k=1}^{n}\left[\binom{n}{k}+\binom{n}{k-1}\right] f^{(k)}(x) + \binom{n+1}{n+1} f^{(n+1)}(x)\right]$
$= e^x\left[\binom{n+1}{0} f^{(0)}(x) + \sum_{k=1}^{n}\binom{n+1}{k} f^{(k)}(x) + \binom{n+1}{n+1} f^{(n+1)}(x)\right]$
$= e^x \sum_{k=0}^{n+1}\binom{n+1}{k} f^{(k)}(x)$

$\therefore$ It is also true for $n = n+1$

$\therefore f^{(n+1)}(x) = e^x \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x)$ for all $n \geq 0 \in \mathbb{Z}^+$

46. Claim $f^{(n)}(0) = B_n$

For $n=0$, $f^{(0)}(0) = e^{e^0-1} = 1 = B_0$

Assume $f^{(k)}(x) = B_k$, for $k = 0, 1, 2, \ldots, n$

$f^{(n+1)}(0) = e^0 \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(0)$
$= \sum_{k=0}^{n}\binom{n}{k} B_k$
$= B_{n+1}$ (Q44)

$\therefore f^{(n)}(0) = B_n$

$e^{e^x-1} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ (Taylor Series)
$= \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$