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Math 475
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42 ok
43 ok

42) For $k \geq 1$

define $g_k(x) = \sum_{n=k}^{\infty} S(n,k) x^n$

Prove: $g_k(x) = x(g_{k-1}(x) + k g_k(x))$

Pf $x \cdot (g_{k-1}(x) + k\, g_k(x))$

$$= x \cdot \left[\sum_{n=k-1}^{\infty} S(n,k-1) x^n + k \cdot \sum_{n=k}^{\infty} S(n,k) x^n \right]$$

$$= x \cdot \left[\underbrace{S(k-1,k-1)}_{=1} x^{k-1} + \sum_{n=k}^{\infty} S(n,k-1) x^n + \sum_{n=k}^{\infty} k \cdot S(n,k) x^n \right]$$

$$= x \cdot \left\{ 1 \cdot x^{k-1} + \left(\sum_{n=k}^{\infty} \left[\underbrace{S(n,k-1) + k \cdot S(n,k)}_{= S(n+1,k)} \right] x^n \right) \right\}$$

$$= x \cdot \left[x^{k-1} + \sum_{n=k}^{\infty} S(n+1,k) x^n \right]$$

$$= x^k + x \cdot \sum_{n=k}^{\infty} S(n+1,k) x^n$$

$$= \underbrace{1}_{= S(k,k)} \cdot x^k + \sum_{n=k}^{\infty} \left(S(n+1,k) x^n \cdot x \right)$$

$$= S(k,k) x^k + \sum_{n=k}^{\infty} S(n+1,k) x^{n+1}$$

$$= S((k-1)+1,k) x^{(k-1)+1} + \sum_{n=k}^{\infty} S(n+1,k) x^{n+1}$$

$$= \sum_{n=k-1}^{\infty} S(n+1,k) x^{n+1}$$

$$\overset{\text{Reindexing}}{=} \sum_{n=k}^{\infty} S(n,k) x^n$$

$$= g_k(x)$$

∎

43) Prove $g_k(x) = \dfrac{x^k}{(1-x)(1-2x)(1-3x)\cdots(1-kx)}$

$k \geq 1$

$k-1 \geq 0$

$k-1 \geq 1$

Pf By Question 42,

$$g_k(x) = x \cdot \left(g_{k-1}(x) + k\, g_k(x)\right)$$

$$g_k(x) = x \cdot g_{k-1}(x) + kx \cdot g_k(x)$$

$$g_k(x) - kx \cdot g_k(x) = x \cdot g_{k-1}(x)$$

$$(1-kx) \cdot g_k(x) = x \cdot g_{k-1}(x)$$

$$\boxed{g_k(x) = \left(\frac{x}{1-kx}\right) \cdot g_{k-1}(x)} \leftarrow \text{The recurrence relation}$$

$$g_k(x) = \left(\frac{x}{1-kx}\right) \cdot g_{k-1}(x)$$

$$= \left(\frac{x}{1-kx}\right) \cdot \left(\frac{x}{1-(k-1)x}\right) \cdot g_{k-2}(x)$$

$$= \left(\frac{x}{1-kx}\right) \cdot \left(\frac{x}{1-(k-1)x}\right) \cdot \left(\frac{x}{1-(k-2)x}\right) \cdot g_{k-3}(x)$$

We have to stop at $g_1(x)$ b/cos from Question 4 $k \geq 1$

$$= \underbrace{\left(\frac{x}{1-kx}\right) \cdot \left(\frac{x}{1-(k-1)x}\right) \cdot \left(\frac{x}{1-(k-2)x}\right) \cdot \left(\frac{x}{1-(k-3)x}\right) \cdots \left(\frac{1}{1-2x}\right)}_{(k-1)\text{ many terms}} g_1(x)$$

$$= \frac{x^{k-1}}{(1-2x)(1-3x)\cdots(1-kx)} \cdot \sum_{n=1}^{\infty} \underbrace{S(n,1)}_{1} x^n$$

$$= \frac{x^{k-1}}{(1-2x)(1-3x)\cdots(1-kx)} \cdot \sum_{n=1}^{\infty} x^n$$

reindexing

$$= \frac{x^{k-1}}{(1-2x)(1-3x)\cdots(1-kx)} \cdot \sum_{n=0}^{\infty} x^{n+1}$$

$$= \frac{x^{k-1}}{(1-2x)(1-3x)\cdots(1-kx)} \cdot x \cdot \sum_{n=0}^{\infty} x^n$$

$$= \frac{x^{k-1} \cdot x}{(1-2x)(1-3x)\cdots(1-kx)} \cdot \left(\frac{1}{1-x}\right)$$

Since geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$$= \frac{x^k}{(1-x)(1-2x)(1-3x)\cdots(1-kx)}$$

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