Steve Homer
Math 475

OK

Homework assigned 3/22

36) Let $a_n = |\{(i,j) : i + 2j = n, \ i,j \geq 0\}|$ and
$g(x) = \sum_{n=0}^{\infty} a_n x^n$

a) Show that $g(x)$ is a rational function of x, i.e. the ratio of polynomials

If the a_n's satisfy a recurrence relation, then $g(x)$ is a rational function.

$$g(x) = (\underbrace{1 + x + x^2 + x^3 + \dots}_{i})(\underbrace{1 + x^2 + x^4 + x^6 + \dots}_{j})$$

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

$$1 + x^2 + x^4 + x^6 + \dots = \frac{1}{1-x^2} \qquad g(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)$$

$$g(x) = \frac{1}{1-x-x^2+x^3} \rightarrow \frac{p(x)}{q(x)} \quad \therefore \text{ rational function}$$

b) $$g(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right) = \left(\frac{1}{1-x}\right)^2\left(\frac{1}{1+x}\right) = \frac{1}{(1+x)(1-x)^2}$$

$$g(x) = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x}$$

$$1 = A - Ax^2 + B + Bx + C - 2Cx + Cx^2$$

$$1 = (A+B+C) + (B-2C)x + (-A+C)x^2$$

$A = C \qquad 2A + B = 1 \qquad 2B = 1$

$\qquad -2A + B = 0 \qquad B = \frac{1}{2} \qquad A = C = \frac{1}{4}$

$$g(x) = \frac{\frac{1}{4}}{1-x} + \frac{\frac{1}{2}}{(1-x)^2} + \frac{\frac{1}{4}}{1+x} = \frac{1}{4}\sum x^n + \frac{1}{2}\sum \binom{n+1}{1} x^n + \frac{1}{4}\sum (-1)^n x^n$$

$$\sum a_n x^n = \sum \left(\frac{1}{4} + \frac{(-1)^n}{4} + \frac{1}{4}(2n+2)\right) x^n \qquad \boxed{a_n = \frac{1}{4}\left(2n+3+(-1)^n\right)}$$

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Alex Fu

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4/5/2010 M

37. Prove $\sum_{k=1}^{m}\binom{m}{k}k^{m-1}(-1)^k=0$

A. Let $f_0(x)=(1+x)^m$,

$f_1(x)=x f_0^{(1)}(x)$,

$f_2(x)=x f_1^{(1)}(x)$, ...

$f_0(x)$: $(1+x)^m=\sum_{k=0}^{m}\binom{m}{k}x^k$

$LHS=(1+x)^m=(1+x)^m g_0(x)$ for some poly $g_0(x)=0$

$RHS=\sum_{k=0}^{m}\binom{m}{k}x^k=\sum_{k=0}^{m}\binom{m}{k}k^0x^k$

$f_1(x)$: $LHS=x\cdot\frac{d}{dx}(1+x)^m g_0(x)=x(1+x)^{m-1}[mg_0(x)+g_0^{(1)}(x)(1+x)]=(1+x)^{m-1}g_1(x)$ for some poly $g_1(x)$

$RHS=x\cdot\frac{d}{dx}\sum_{k=0}^{m}\binom{m}{k}x^k=\sum_{k=1}^{m}\binom{m}{k}k^1x^k$

$f_2(x)$: $LHS=x\cdot\frac{d}{dx}(1+x)^{m-1}g_1(x)=x(1+x)^{m-2}[(m-1)g_1(x)+g_1^{(1)}(x)(1+x)]=(1+x)^{m-2}g_2(x)$ for some poly $g_2(x)$

$RHS=x\cdot\frac{d}{dx}\sum_{k=1}^{m}\binom{m}{k}k^1x^k=\sum_{k=1}^{m}\binom{m}{k}k^2x^k$

⋮

$f_{m-1}(x)$: $LHS=(1+x)^{m-(m-1)}g_{m-1}(x)=(1+x)g_{m-1}(x)$ for some poly $g_{m-1}(x)$

$RHS=\sum_{k=1}^{m}\binom{m}{k}k^{m-1}x^k$

$f_{m-1}(-1)=[1+(-1)]\,g_{m-1}(-1)=0$

$\therefore\ 0=\sum_{k=1}^{m}\binom{m}{k}k^{m-1}(-1)^k$