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Homework 19

OK

(33) Given the recurrence relation $a_{n+1} = a_n + 6a_{n-1}$.
(a) Find r, s s.t. $a_n = \alpha r^n + \beta s^n$ is the general solution.
(b) Given $a_0 = 1$ & $a_1 = 2$ find α, β.
(c) Given $a_0 = 2$ & $a_1 = 1$ find α, β.

(a) Given $a_{n+1} = a_n + 6a_{n-1}$, $x^2 = x + 6$

$$x^2 - x - 6 = 0$$
$$(x-3)(x+2)$$
$$x = \underset{r}{3}, \underset{s}{-2}$$

So the general solution is, $a_n = \alpha(3^n) + \beta(-2^n)$

(b) Given $a_0 = 1$, $a_1 = 2$

$a_0 = \alpha + \beta = 1 \Rightarrow \alpha = 1 - \beta$

$a_1 = 3\alpha - 2\beta = 2 \Rightarrow 3 - 3\beta - 2\beta = 2$

$$3 - 5\beta = 2$$
$$-5\beta = -1$$
$$\boxed{\beta = \frac{1}{5}, \quad \alpha = \frac{4}{5}}$$

(c) Given $a_0 = 2$, $a_1 = 1$

$a_0 = \alpha + \beta = 2 \Rightarrow \alpha = 2 - \beta$

$a_1 = 3\alpha - 2\beta = 1 \Rightarrow 6 - 3\beta - 2\beta = 1$

$$-5\beta = -5$$
$$\boxed{\beta = 1, \quad \alpha = 1}$$

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(34) Given the recurrence relation $a_{n+1} = a_n + 6a_{n-1}$ let

$g(x) = a_0 + a_1 x + a_2 x^2 + \dots$

be ~~a~~ the associated generating function. Find polynomials p & q with $g(x) = \frac{p(x)}{q(x)}$.

We will use the fact that $a_1 + 6a_0 = a_2$.

$g(x) = a_0 + a_1 x + a_2 x^2 + \dots$

(1) $x g(x) - a_0 x = a_1 x^2 + a_2 x^3 + \dots$

(2) $6x^2 g(x) = 6a_0 x^2 + 6a_1 x^3 + \dots$

(3) $g(x) - (a_0 + a_1 x) = a_2 x^2 + a_3 x^3 + \dots$

Notice (1) + (2) = (3) since $a_1 + 6a_0 = a_2$, & $a_2 + 6a_1 = a_3, \dots$

$$x g(x) - a_0 x + 6x^2 g(x) = g(x) - a_0 - a_1 x$$

$$-g(x) + x g(x) + 6x^2 g(x) = a_0 x - a_1 x - a_0$$

$$g(x)\left(-1 + x + 6x^2\right) = (a_0 - a_1)x - a_0$$

$$\boxed{g(x) = \frac{(a_0 - a_1)x - a_0}{(6x^2 + x - 1)}} \quad \begin{matrix} = p(x) \\ = q(x) \end{matrix}$$

35 Given $\frac{x+1}{x^2+2x-3} = a_0 + a_1 x + a_2 x^2 + \cdots$
find constants p, q s.t. for every $n = 1, 2, \ldots$
$$a_{n+1} = p a_n + q a_{n-1}$$

$$\frac{x+1}{x^2+2x-3} = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$x+1 = (x^2+2x-3)(a_0 + a_1 x + a_2 x^2 + \cdots)$$
$$x+1 = -3a_0 + (2a_0 - 3a_1)x + (a_0 + 2a_1 + 3a_2)x^2 + (a_1 + 2a_2 - 3a_3)x^3 + \cdots$$

Then,
$$0 = a_0 + 2a_1 - 3a_2 = a_1 + 2a_2 - 3a_3 = \cdots$$
Note,
$$0 = a_{n-1} + 2a_n - 3a_{n+1}$$
$$3a_{n+1} = a_{n-1} + 2a_n$$
$$\boxed{a_{n+1} = \tfrac{2}{3}a_n + \tfrac{1}{3}a_{n-1}, \; p = \tfrac{2}{3}, \; q = \tfrac{1}{3}}$$

Alternatively, by looking at x^2+2x-3, it is clear by problems 33,34 that ~~these~~ numbers are the steps to determine $g(x)$.
So, $x^2+2x-3=0$ translates to $0 = a_{n-1} + 2a_n - 3a_{n+1}$
$$\Rightarrow 3a_{n+1} = a_{n-1}\,2a_n$$
$$\Rightarrow \boxed{a_{n+1} = \tfrac{2}{3}a_n + \tfrac{1}{3}a_{n-1}}$$
which is the same answer.