Maheswaran
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CS 475
(3-12 F)

29 ok
30 ok

29. 6 men are randomly returned their 6 hats. Find the following probabilities:

(a) No man gets his own hat

This is nothing but derangement of a set of 6 elements / 6!

$\therefore$ Answer $= \frac{D_6}{6!} = \frac{265}{720} = 0.3685$

(b) Every man gets his own hat.

No. of ways for doing this is 1 [Everybody gets their own hat → only one possible way]

Total no. of ways $= 6! = 720$.

$\therefore$ Prob $= 1/6! = 1/720$

(c) Exactly one man gets his own hat.

This person could be chosen in $\binom{6}{1}$ ways. All the remaining 5 should not get their own hat $\Rightarrow D_5$

$\therefore$ Total No. of ways that exactly one person gets his own hat $\left.\right\}$ $= \binom{6}{1} D_5$

$= 6 \times 44$

$= 264$

$$\therefore \text{Prob} = \frac{264}{6!} = \frac{264}{720}$$

(d) At least 2 men get their own hat.

= All possible combinations − No man gets his own hat − Exactly one man gets his own hat

$$= \frac{6! - D_6 - 6D_5}{6!}$$

$$= \frac{6! - 265 - 264}{6!} = \frac{191}{720}$$

(e) for any n,

(a) $\rightarrow \dfrac{D_n}{n!}$ (c) $\dfrac{nD_n}{n!}$

(b) $\rightarrow \dfrac{1}{n!}$ (d) $\dfrac{n! - D_n - nD_{n-1}}{n!}$

30. How many permutations of 1,2,3,4,5,6,7 are there which do not contain any block of the form 123, 345 or 567?

No. of All possible permutations of 1,2,..7 is $7! = |S|$

$A_1 \rightarrow$ set containing the block 123

$|A_1| = 5!$

$A_2 \rightarrow$ set containing the block 345

$|A_2| = 5!$

$A_3 \rightarrow$ set containing the block 567

$|A_3| = 5!$

$|A_1 \cap A_2| = 3!$ $\quad [\boxed{12345}\ 6\ 7]$

$|A_2 \cap A_3| = 3!$

$|A_2 \cap A_3| = 3!$

$|A_1 \cap A_2 \cap A_3| = 1!$ $\quad [\boxed{1\ 2\ 3\ 4\ 5\ 6\ 7}]$

$\therefore$ Answer $= 7! - 3\times 5! + 3\times 3! - 1!$ $\quad [\,|S| - |A_1 \cup A_2 \cup A_3|\,]$

4697

$\boxed{\text{Answer} = \cancel{4661}}$

ADAM WRIGHT

MONDAY 22nd

(31) $\mathcal{G}_n$ set of all $X \subseteq \{1,2,\dots,n\}$ w/ no adjacent integers & $g_n = |\mathcal{G}_n|$

- $\mathcal{G}_0 = \{\emptyset\}$, so $g_0 = 1$ — 31 ok
- $\mathcal{G}_1 = \{\emptyset, \{1\}\}$, so $g_1 = 2$ — 32 ok
- $\mathcal{G}_2 = \{\emptyset, \{1\}, \{2\}\}$, so $g_2 = 3$
- $\mathcal{G}_3 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}\}$, so $g_3 = 5$

Pf: $g_{n+1} = g_n + g_{n-1}$

Note: Each set G_n will contain all the elements of G_m (where $m > n$).

- The set G_{n+1} is obtained by including all elements of G_n, & then appending n to all elements in G_n which do not contain the element n in them (because you cannot contain 2 consecutive intergers). All the elements in G_n that do not contain n in them is exactly the same as the set G_{n-1}. Thus G_{n+1} is the all elements of G_n, and all elements of G_{n-1} w/ $n+1$ appended to each elements, which I will refer to as $G^{\star}_{n-1}$. So $G_{n+1} = G^{\star}_{n-1} \cup G_n$ & G_n and $G^{\star}_{n-1}$ are disjoint (because all elements of $G^{\star}_{n-1}$ contain $n+1$, & no elements of G_n contain $n+1$). So $|G_{n+1}| = |G^{\star}_{n-1}| + |G_n|$, & since $G^{\star}_{n-1}$ contain all the elements of G_{n-1} w/ $n+1$ appended to all the elements, $|G^{\star}_{n-1}| = |G_{n-1}|$, therefor:

$$|G_{n+1}| = |G_{n-1}| + |G_n|, \text{ i.e. } \boxed{g_{n+1} = g_{n-1} + g_n} \blacksquare$$

- also $g_1 = 2$ & $g_2 = 3$ (above), so $g_1 = f_3$ & $g_2 = f_4$, (f_n is the nth element of the fibinachi sequence). & Since the g_n's & the f_n's satisfy the same recurrence relation ($f_{n+1} = f_{n-1} + f_n$), using induction we see that $g_n = f_{n+2}$ for $n=1$ ($g_1 = f_3 = 2$), & if we assume its true for all integers up to n. then $g_{n+1} = f_{n+3}$ simplifies to $g_{n-1} + g_n = f_{n+2} + f_{n+1}$ (because $g_{n+1} = g_{n-1} + g_n$ & $f_{n+3} = f_{n+2} + f_{n+1}$), so because $g_{n-1} = f_{n+1}$ & $g_n = f_{n+2}$, this is clearly true. Therefor $\boxed{g_n = f_{n+2} \text{ for all } n \geq 1}$. $\blacksquare$

(32) Let $g_{n,k} = |\{X \in G_n : |X| = k\}|$ + Prove $g_{n,k} = \binom{n-k+1}{k}$.

- Lets relate $g_{n,k}$ to the number of possible of possible bit strings of length n, which contain exactly k ones, + there is always a zero between any two ones. This is equivalent to the number of bit strings of length n w/ exactly $n-(k-1)$ ones + no restrictions on the placement of the ones (get rid of restriction by automatically putting a zero in the $k-1$ spaces between the k ones). This set can be counted by choosing k of the $n-k+1$ spots. Therefore:

$$g_{n,k} = \binom{n-k+1}{k} \blacksquare$$

- g_n is simply $g_{n,k}$ for all possible values of k. Using the bit string analogy g_n is like bit strings of length $n-k+1$, write k ones, where k can be any number $1,\ldots,n$. So $g_n = \sum_{k=0}^{n} \binom{n-k+1}{k}$, where if $n-k+1 < k$, then $\binom{n-k+1}{k} = 0$. Since $g_n = f_{n+2}$:

$$\boxed{f_{n+2} = \sum_{k=0}^{n} \binom{n-k+1}{k}}, \text{ where if } n-k+1<k, \text{ then } \binom{n-k+1}{k}=0$$