27 ok
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Math 475
03/17/2010

27) How many ways can you put 20 indistinguishable balls into 4 distinguishable buckets so that every bucket has at least one ball and no ~~bucket~~ has more than 7 balls?

We put 1 ball into each bucket, thus this question is reduced to putting 16 indistinguishable balls into 4 distinguishable buckets so ~~that~~ that we are finding the size of

$$\{ \langle x_1, x_2, x_3, x_4 \rangle : x_1 + x_2 + x_3 + x_4 = 16$$

$$0 \le x_1 \le 6$$

and $0 \le x_2 \le 6$

$$0 \le x_3 \le 6$$

$$0 \le x_4 \le 6 \quad \}$$

$$Q = \{ \langle x_1, x_2, x_3, x_4 \rangle : x_1 + x_2 + x_3 + x_4 = 16 \}$$

$$|Q| = \binom{16+3}{3} = \cancel{\binom{18}{3}} \binom{19}{3} = \frac{19!}{3!\,16!} = \frac{19 \cdot \cancel{18}^{3} \cdot 17}{\cancel{3} \cdot \cancel{2} \cdot 1} = 969 = \binom{19}{3}$$

$$A_1 = \{ \langle x_1, x_2, x_3, x_4 \rangle \in Q : x_1 \ge 7 \}$$

$$A_2 = \{ \langle x_1, x_2, x_3, x_4 \rangle \in Q : x_2 \ge 7 \}$$

$$A_3 = \{ \langle x_1, x_2, x_3, x_4 \rangle \in Q : x_3 \ge 7 \}$$

$$A_4 = \{ \langle x_1, x_2, x_3, x_4 \rangle \in Q : x_4 \ge 7 \}$$

① $A_1 = \{\langle x_1, x_2, x_3, x_4 \rangle \in Q : x_1 \geq 7\}$

$y_1 = x_1 - 7, \quad y_1 + x_2 + x_3 + x_4 = 9$

$|A_1| = \binom{9+3}{3} = \binom{12}{3} = \frac{12!}{3!\,9!} = \frac{\overset{2}{\cancel{12}} \cdot 11 \cdot 10}{\cancel{3} \cdot \cancel{2} \cdot 1} = 220$

Similarly, $|A_2| = |A_3| = |A_4| = |A_1| = 220 = \binom{12}{3}$

② $A_1 \cap A_2 = \{\langle x_1, x_2, x_3, x_4 \rangle \in Q : x_1 \geq 7 \text{ and } x_2 \geq 7\}$

$y_1 = x_1 - 7, \quad y_2 = x_2 - 7, \quad y_1 + y_2 + x_3 + x_4 = 2$

$|A_1 \cap A_2| = \binom{2+3}{3} = \binom{5}{3} = \frac{5!}{2!\,3!} = \frac{5 \cdot \overset{2}{\cancel{4}}}{\cancel{2} \cdot 1} = 10$

Similarly, $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4|$
$= |A_3 \cap A_4| = 10 = \binom{5}{3}$

③ $A_1 \cap A_2 \cap A_3 = \{\langle x_1, x_2, x_3, x_4 \rangle \in Q : x_1 \geq 7 \text{ and } x_2 \geq 7, \text{and } x_3 \geq 7\}$

This is an impossible case, thus $|A_1 \cap A_2 \cap A_3| = 0$

Similarly, $|A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = |A_1 \cap A_2 \cap A_3| = 0$

④ $A_1 \cap A_2 \cap A_3 \cap A_4 = \{\langle x_1, x_2, x_3, x_4 \rangle \in Q : x_1 \geq 7 \text{ and } x_2 \geq 7 \text{ and } x_3 \geq 7 \text{ and } x_4 \geq 7\}$

This is an impossible case, thus $|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$

Hence $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |Q - (A_1 \cup A_2 \cup A_3)|$

$= |Q| - |A_1 \cup A_2 \cup A_3|$

$= |Q| - |A_1| - |A_2| - |A_3| - |A_4| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4|$
$+ |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3| - |A_1 \cap A_2 \cap A_4|$
$- |A_1 \cap A_3 \cap A_4| - |A_2 \cap A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3 \cap A_4|$

$= \binom{4}{0}\binom{19}{3} - \binom{4}{1} \cdot \binom{12}{3} + \binom{4}{2} \cdot \binom{5}{3} \mp \binom{4}{3} \cdot 0 + \binom{4}{4} \cdot 0$

$= 1 \cdot 969 - 4 \cdot 220 + 6 \cdot 10 \mp 4 \cdot 0 + 1 \cdot 0 = \boxed{149}$

28) How many ways can you put 5 red balls and 5 blue balls into 3 distinct boxes so that no box is empty?

$|Q|$ = # of ways to put 5 red balls and 5 blue balls into 3 distinct boxes

$$= \binom{5+2}{2} \times \binom{5+2}{2} = \binom{7}{2} \times \binom{7}{2} = \frac{7!}{2!5!} \times \frac{7!}{2!5!} = 21 \times 21 = 441$$

~~$A_i = \{\text{Box } i \text{ empty} : i = 1, 2, 3\}$~~

For $i = 1, 2, 3$,

$A_i = \{\text{Box } i \text{ empty}\} \rightsquigarrow$ means put the 5 red balls and 5 blue balls into the 2 distinct nonempty boxes

$$|A_1| = |A_2| = |A_3| = \binom{5+1}{1} \times \binom{5+1}{1} = \binom{6}{1} \times \binom{6}{1} = 6 \times 6 = 36$$

For $i \neq j$, $1 \leq i \leq 3$, $1 \leq j \leq 3$,

$A_i \cap A_j = \{\text{Box } i \text{ and box } j \text{ both empty}\} \rightsquigarrow$ means put 5 red balls and 5 blue balls into the only one nonempty box

$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = 1 \rightsquigarrow$ because only one way to put all balls into 1 box.

$A_1 \cap A_2 \cap A_3 = \{\text{all 3 boxes empty}\}$

$|A_1 \cap A_2 \cap A_3| = 0$

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |Q - (A_1 \cup A_2 \cup A_3)|$$

$$= |Q| - |A_1 \cup A_2 \cup A_3|$$

$$= |Q| - |A_1| - |A_2| - |A_3| + |A_1 \cup A_2| + |A_1 \cup A_3| + |A_2 \cup A_3| - |A_1 \cup A_2 \cup A_3|$$

$$= \binom{3}{0} \times \binom{7}{2} \times \binom{7}{2} - \binom{3}{1} \times \binom{6}{1} \times \binom{6}{1} + \binom{3}{2} \times 1 - \binom{3}{3} \times 0$$

$$= 1 \times 441 - 3 \times 36 + 3 \times 1 - 1 \times 0 = \boxed{336}$$