Math 475 HW

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OK

(25) Prove:

$$\binom{3n}{n} = \sum_{a+b+c=n} \binom{n}{a}\binom{n}{b}\binom{n}{c}$$

Let S be a set with $3n$ elements.
Partition S into 3 subsets, A, B, C, each with n elements
To make an n-sized subset of S, a elements are chosen from A, b elements are chosen from B, and c elements chosen from C. $(c = n-a-b)$.
To count the number of these n-sized subsets, we can count the number of ways to choose a elements from A $(a \leq n)$, then multiply it the number of ways to choose b elements from B $(b \leq n-a)$, and finally multiply this by the number of ways to choose c elements from C $(c = n-a-b)$.
This must be repeated for all the ways that a, b, and c can be chosen.

So $\binom{n}{0}\binom{n}{0}\binom{n}{4} + \binom{n}{0}\binom{n}{1}\binom{n}{3} + \ldots + \binom{n}{3}\binom{n}{1}\binom{n}{0} + \ldots + \binom{n}{4}\binom{n}{0}\binom{n}{0}$

This is generalized to:

$$\sum_{a+b+c=n}^{n} \binom{n}{a}\binom{n}{b}\binom{n}{c}$$