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Ok

2/17/2010 W

M475

24. Prove $\dfrac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n} \dfrac{1}{k+1}\binom{n}{k}$

A: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

1° Put $y=1$: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

2° $\int (1+x)^n dx = \int \sum_{k=0}^{n} \binom{n}{k} x^k \, dx$

$\dfrac{(1+x)^{n+1}}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \int x^k dx$

$= \sum_{k=0}^{n} \binom{n}{k} \cdot \dfrac{x^{k+1}}{k+1} + C \quad , \exists C \in N$

3° Put $x=0$: $\dfrac{1}{n+1} = C$

4° Put $x=1$: $\dfrac{(1+1)^{n+1}}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \dfrac{1}{k+1} + \dfrac{1}{n+1}$

$\dfrac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n} \dfrac{1}{k+1} \binom{n}{k}$

Math 475 HW 24

Sean Bauer

OK

24) Prove:

$$\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$$

Start w/ 2^{n+1}

$$2^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} = \sum_{k=0}^{n+1} \frac{(n+1)!}{(n+1-k)!\,k!}$$

Make sum just to n, take out $k=n+1$ $\binom{n+1}{n+1} = 1$

$$= \sum_{k=0}^{n} \frac{(n+1)!}{(n+1-k)!\,k!} + 1$$

$$2^{n+1} - 1 = \sum_{k=0}^{n} \left(\frac{n+1!}{(n+1-k)!\,k!}\right) = \sum_{k=0}^{n} \frac{(n+1)\,n!}{(n+1-k)(n-k)!\,k!} = (n+1)\sum_{k=0}^{n} \frac{1}{n+1-k}\binom{n}{k}$$

so $$\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n} \frac{1}{(1+n-k)}\binom{n}{k}.$$ Recall $\binom{n}{k} = \binom{n}{n-k}$

$$= \sum_{k=0}^{n} \frac{1}{(n-k+1)}\binom{n}{n-k}$$ Set $j = n-k$, and rewrite sum.

$$= \sum_{j=n}^{0} \frac{1}{j+1}\binom{n}{j}$$ can flip limits, change j back to k.

$\therefore$ $$\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$$ ✓