

Vector Spaces

A vector space, V , is a set with two operations, vector addition (written $u + v$) and scalar multiplication (written av). The elements of V will be denoted using u, v, w , etc. The formula ' $u \in V$ ' is short hand for ' u is an element of V ' or ' u in V ' or just ' u is a vector'. Vector spaces will be written using capital letters V, W , etc. Scalars are elements of some field, which in this class is either the real numbers, $\mathbb{R}$, or the complex numbers, $\mathbb{C}$. Scalars will be written using the letters a, b, c , etc.

Closure axioms:

1. If $u \in V$ and $v \in V$, then $u + v \in V$.
2. If $u \in V$ and a a scalar, then $au \in V$.

Associative, commutative, distributive axioms:

1. For all $u, v, w \in V$ $(u + v) + w = u + (v + w)$.
2. For all $u, v \in V$ $u + v = v + u$.
3. For all scalars a and b and vectors $u \in V$ $(ab)u = a(bu)$.
4. For all scalars a and b and vectors $u \in V$ $(a + b)u = au + bu$.
5. For all scalars a and vectors $u, v \in V$ $a(u + v) = au + av$.

Zero vector, additive inverse, identity axioms:

1. There exists a vector $z \in V$ such that for all $u \in V$ $z + u = u + z = u$.
2. For every $u \in V$ there exists a vector $v \in V$ (for which we write $v = -u$) such that $u + v = v + u = z$.
3. For every $u \in V$, $1u = u$.

Any abstract set V with two operations, vector addition and scalar multiplication which satisfy all the above axioms is a vector space.

Most author's use either $\mathbf{0}$ or $\vec{0}$ to denote the zero vector but students persistently confuse the zero vector with the zero scalar, so I decided to write the zero vector as z .

Exercise 1 *Prove that $0u = z$ for any $u \in V$ a vector space.*

Exercise 2 *Prove that $(-1)u = -u$ for any $u \in V$ a vector space.*

Definition 3 For W a subset of a vector space V (written $W \subseteq V$) we say that W is a subspace of V iff

1. for every $u, v \in V$ if $u \in W$ and $v \in W$, then $u + v \in W$, and
2. for every $u \in V$ and scalar a if $u \in W$, then $au \in W$.

I usually add to the definition that W be a nonempty set, but this semester I have decided not to worry about the empty set or should I say I have no worries? or should I say my set of worries is the empty set?

Theorem 4 *If W is a subspace of V , then W is itself a vector space under the operations defined in V .*

proof:

The closure axioms are easy since they are practically the same as the definition of subspace. The associative, commutative, distributive axioms are true in W because they are true in V and W is a subset of V . The zero vector z is in W because $0u = z$ (exercise 1) so (assuming W is nonempty) if anything is in W , then z is in W . Similarly $(-1)u = -u$ (exercise 2), so if $u \in W$, then also $-u \in W$.

QED

Theorem 5 *Suppose W is a subset of V (i.e., $W \subseteq V$). Then*

1. W is a subspace of V
iff
2. for every $u, v \in W$ and scalars a, b we have $au + bv \in W$.

proof:

(1) implies (2):

Assume W is a subspace of V . Suppose $u, v \in W$ and a, b are scalars. By the second axiom of subspaces we have that $au \in W$ and $bv \in W$. Letting $w_1 = au$ and $w_2 = bv$ we have that $w_1 \in W$ and $w_2 \in W$, therefore by the first axiom of subspaces we have that $w_1 + w_2 \in W$ and so $au + bv \in W$.

(2) implies (1):

Assume (2): for every $u, v \in W$ and scalars a, b we have $au + bv \in W$. We must show the two axioms of a subspace hold for W . Suppose $u, v \in W$. Then letting $a = b = 1$ we have that $1u + 1v \in W$, so $1u + 1v = u + v \in W$. For the second axiom, suppose $u \in W$ and a any scalar, then we have that $au + 0u \in W$ by (2), but $au = au + 0u = au + 0u$ so $au \in W$.

QED

Definition 6 For $u_1, \dots, u_n$ elements of a vector space V , define their span:

$$\text{span}(\{u_1, u_2, \dots, u_n\}) = \{a_1u_1 + a_2u_2 + \dots + a_nu_n : a_1, a_2, \dots, a_n \text{ scalars}\}$$

Each of the vectors $a_1u_1 + a_2u_2 + \dots + a_nu_n$ is called a linear combination of the u 's so we could also say that the span is the set of all linear combinations. If $W = \text{span}(\{u_1, u_2, \dots, u_n\})$, we say that ' W is spanned by $u_1, u_2, \dots, u_n$ ' or ' $u_1, u_2, \dots, u_n$ span W '. The closure axioms of a vector space V guarantee that if $u_1, u_2, \dots, u_n \in V$, then $\text{span}(\{u_1, u_2, \dots, u_n\}) \subseteq V$. This is true because the second closure axiom says each a_iu_i is in V , while the first axiom guarantees that their sum is in V , e.g., if we write $v_i = a_iu_i$, $v_1, v_2 \in V$ implies $v_1 + v_2 \in V$ and so $v_1 + v_2, v_3 \in V$ implies $v_1 + v_2 + v_3 = (v_1 + v_2) + v_3 \in V$, and so on.

Theorem 7 Suppose $u_1, u_2, \dots, u_n$ are elements of W which is a subspace of V . Then $\text{span}(\{u_1, u_2, \dots, u_n\}) \subseteq W$.

proof:

Suppose $v \in \text{span}(\{u_1, u_2, \dots, u_n\})$. Then for some scalars, $a_1, \dots, a_n$ we have that

$$v = a_1u_1 + \dots + a_nu_n.$$

Since W is a subspace of V we have that $a_iu_i \in W$ for each i . Now let $v_i = a_iu_i$ to simplify our writing. Since $v_1 \in W$ and $v_2 \in W$ we have by the

first axiom of subspaces that $v_1 + v_2 \in W$. Thus we have that the two vectors $v_1 + v_2$ and v_3 are elements of W . This means their sum $(v_1 + v_2) + v_3$ is in W . Continuing on like this we see that $v_1 + v_2 + \cdots + v_k \in W$ for each k and so

$$v = a_1u_1 + \cdots + a_nu_n = v_1 + v_2 + \cdots + v_n \in W$$

as we needed to show.

QED

Theorem 8 *Suppose $u_1, u_2, \dots, u_n$ are elements of a vector space V . Then $\text{span}(\{u_1, u_2, \dots, u_n\})$ is a subspace of V .*

proof:

We verify each of the axioms of a subspace. Let

$$W = \text{span}(\{u_1, u_2, \dots, u_n\}).$$

Suppose v, w are elements of W . Then since W is the span of the u 's there exists scalars $c_1, \dots, c_n$ and $d_1, \dots, d_n$ such that

$$v = c_1u_1 + \cdots + c_nu_n \text{ and } w = d_1u_1 + \cdots + d_nu_n.$$

But then

$$v + w = (c_1 + d_1)u_1 + (c_2 + d_2)u_2 + \cdots + (c_n + d_n)u_n$$

and so $v + w \in \text{span}(\{u_1, u_2, \dots, u_n\}) = W$

For the second axiom, suppose $v \in W$ and a a scalar. Then for some scalars $c_1, \dots, c_n$

$$v = c_1u_1 + \cdots + c_nu_n$$

but then

$$av = a(c_1u_1 + \cdots + c_nu_n) = (ac_1)u_1 + \cdots + (ac_n)u_n$$

and so $av \in \text{span}(\{u_1, u_2, \dots, u_n\}) = W$.

QED

It follows from these last two theorems that $\text{span}(\{u_1, u_2, \dots, u_n\})$ is the smallest subspace of V which contains the vectors $u_1, u_2, \dots, u_n$.

Theorem 9 Suppose $u \in \text{span}(\{u_1, \dots, u_n\})$ then

$$\text{span}(\{u, u_1, \dots, u_n\}) = \text{span}(\{u_1, \dots, u_n\})$$

proof:

To show two sets A and B are equal, $A = B$,
show that
 $A \subseteq B$ and $B \subseteq A$.

To show that $A \subseteq B$
suppose that $x \in A$ and then show $x \in B$.

First we show $\text{span}(\{u, u_1, \dots, u_n\}) \subseteq \text{span}(\{u_1, \dots, u_n\})$. Since $u \in \text{span}(\{u_1, \dots, u_n\})$ there exists scalars $b_1, \dots, b_n$ so that $u = b_1u_1 + \dots + b_nu_n$. Now let w be any element of $\text{span}(\{u, u_1, \dots, u_n\})$. This means there are scalars $a, a_1, \dots, a_n$ such that

$$w = au + a_1u_1 + \dots + a_nu_n.$$

But then substituting for u :

$$\begin{aligned} w &= a(b_1u_1 + \dots + b_nu_n) + a_1u_1 + \dots + a_nu_n \\ &= (ab_1 + a_1)u_1 + \dots + (ab_n + a_n)u_n \end{aligned}$$

and so $w \in \text{span}(\{u_1, \dots, u_n\})$ as was to be shown.

Second we show $\text{span}(\{u_1, \dots, u_n\}) \subseteq \text{span}(\{u, u_1, \dots, u_n\})$. This is easier. Suppose $w \in \text{span}(\{u_1, \dots, u_n\})$. Then there are scalars $c_1, \dots, c_n$ so that

$$w = c_1u_1 + \dots + c_nu_n$$

but then

$$w = 0u + c_1u_1 + \dots + c_nu_n$$

so $w \in \text{span}(\{u, u_1, \dots, u_n\})$ as was to be shown.

QED

Definition 10 For $v_1, v_2, \dots, v_n$ vectors in a vector space V we say that they are linearly independent iff for any scalars $a_1, a_2, \dots, a_n$

$$a_1v_1 + a_2v_2 + \dots + a_nv_n = z \rightarrow a_1 = a_2 = \dots = a_n = 0$$

Definition 11 We say $v_1, v_2, \dots, v_n$ are linearly dependent iff $v_1, \dots, v_n$ are not linearly independent.

Definition 12 $v_1, v_2, \dots, v_n$ is a basis for the vector space V iff

1. $v_1, v_2, \dots, v_n$ are linearly independent and
2. $V = \text{span}(\{v_1, v_2, \dots, v_n\})$.

Theorem 13 Let A be any $n \times n$ matrix. Then A is invertible iff the set of columns of A is a basis for $\mathbb{R}^n$, i.e., $\text{col}_1(A), \text{col}_2(A), \dots, \text{col}_n(A)$ is a basis for $\mathbb{R}^n$.

proof:

Before proving this result we first prove the following Lemma.

Lemma 14 Suppose A is an $m \times n$ matrix and

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

is $n \times 1$. Then

$$AB = A \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = b_1 \text{col}_1(A) + b_2 \text{col}_2(A) + \dots + b_n \text{col}_n(A) = \sum_{k=1}^n b_k \text{col}_k(A)$$

proof:

Write the matrix A as follows:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & & & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{bmatrix}$$

i.e., $a_{i,j} = \text{entry}_{i,j}(A)$. Then

$$\begin{aligned} AB &= \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & & & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\ &= \begin{bmatrix} a_{1,1}b_1 + a_{1,2}b_2 + a_{1,3}b_3 + \cdots + a_{1,n}b_n \\ a_{2,1}b_1 + a_{2,2}b_2 + a_{2,3}b_3 + \cdots + a_{2,n}b_n \\ a_{3,1}b_1 + a_{3,2}b_2 + a_{3,3}b_3 + \cdots + a_{3,n}b_n \\ \vdots \\ a_{m,1}b_1 + a_{m,2}b_2 + a_{m,3}b_3 + \cdots + a_{m,n}b_n \end{bmatrix}_{m \times 1} \\ &= b_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \\ \vdots \\ a_{m,1} \end{bmatrix} + b_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \\ \vdots \\ a_{m,2} \end{bmatrix} + b_3 \begin{bmatrix} a_{1,3} \\ a_{2,3} \\ a_{3,3} \\ \vdots \\ a_{m,3} \end{bmatrix} + \cdots + b_n \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ a_{3,n} \\ \vdots \\ a_{m,n} \end{bmatrix} \end{aligned}$$

This proves the Lemma.

QED

Now to prove the Theorem we first assume that A is invertible and show that the columns of A are a basis for $\mathbb{R}^n$. To see that they are independent, suppose that

$$b_1 \text{col}_1(A) + \cdots + b_n \text{col}_n(A) = Z$$

where $Z_{n \times 1}$ is the zero vector. By the lemma

$$AB = Z$$

where B is the column vector made from $b_1, \dots, b_n$. Since A is invertible we have that $B = A^{-1}Z = Z$ so $B = Z$ and so $b_i = 0$ for all i with $1 \leq i \leq n$. This shows the the columns of A are linearly independent. To see that they span $\mathbb{R}^n$, let C be an arbitrary element of $\mathbb{R}^n$ so that

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Since A is invertible if we set $B = A^{-1}C$ then we know that $AB = C$ and by the lemma we have that

$$b_1 \text{col}_1(A) + \dots + b_n \text{col}_n(A) = C$$

and so C is in the span of the columns of A . This shows that if A is invertible then its columns are a basis. Next we prove the converse, using the contrapositive.

The contrapositive of the implication:
 P implies Q
 is
 $(\text{Not } Q)$ implies $(\text{Not } P)$
 They are logically equivalent.

Assume that A is not invertible. Then as was shown the algorithm for attempting to invert A produces a $B_{n \times 1} \neq Z$ such that $AB = Z$. This means by the Lemma that

$$b_1 \text{col}_1(A) + \dots + b_n \text{col}_n(A) = Z$$

and since $B \neq Z$ at least one $b_i \neq 0$. But this means that the columns of A are linearly dependent and hence not a basis. This finishes the proof of the Theorem.

QED

Theorem 15 $v_1, v_2, \dots, v_n$ are linearly dependent iff there are scalars

$$a_1, a_2, \dots, a_n$$

such that $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = z$ and for at least one i we have $a_i \neq 0$.

proof:

Not For ALL x Statement(x)
is logically equivalent to
There exists x such that Not Statement(x)

So negating linear independence gives us:

There exists scalars $a_1, \dots, a_n$ such that

Not $[a_1v_1 + \dots + a_nv_n = z \text{ implies } a_1 = a_2 = \dots = a_n = 0]$.

The negation of an implication
Not $[P \text{ implies } Q]$
is logically equivalent to
 P and Not Q .

So in this case we get

$a_1v_1 + \dots + a_nv_n = z$ and Not $[a_1 = a_2 = \dots = a_n = 0]$.

But Not $[a_1 = a_2 = \dots = a_n = 0]$ is the same as saying ‘at least one of the a_i is not equal to 0.

QED

Exercise 16 Prove that $v_1, \dots, v_n$ are linearly dependent iff $v_1 = z$ or $v_j \in \text{span}(\{v_1, \dots, v_{j-1}\})$ for some j with $1 < j \leq n$.

Lemma 17 (Exchange) Suppose $v_1, v_2, \dots, v_{k+1}, w_1, \dots, w_m$ are vectors in a vector space V and

1. $v_1, v_2, \dots, v_{k+1}$ are linearly independent, and
2. $\text{span}(\{v_1, v_2, \dots, v_k, w_1, \dots, w_m\}) = V$.

Then for some i with $1 \leq i \leq m$

$$\text{span}(\{v_1, v_2, \dots, v_{k+1}, w_1, w_2, \dots, w_{i-1}, w_{i+1}, \dots, w_m\}) = V$$

(i.e., we have added v_{k+1} and removed w_i .)

proof:

By (2) there are scalars a_i, b_j such that

$$v_{k+1} = a_1v_1 + a_2v_2 + \cdots + a_kv_k + b_1w_1 + b_2w_2 + \cdots + b_mw_m.$$

It must be that for some i with $1 \leq i \leq m$ that $b_i \neq 0$, because otherwise we would have that

$$v_{k+1} = a_1v_1 + a_2v_2 + \cdots + a_kv_k$$

and therefore

$$z = a_1v_1 + a_2v_2 + \cdots + a_kv_k + (-1)v_{k+1}$$

contradicting their independence (1).

Therefore we have that

$$\begin{aligned} -b_iw_i &= a_1v_1 + \cdots + a_kv_k + (-1)v_{k+1} + \\ &+ b_1w_1 + \cdots + b_{i-1}w_{i-1} + b_{i+1}w_{i+1} + \cdots + b_mw_m \end{aligned}$$

and since $b_i \neq 0$

$$\begin{aligned} w_i &= \frac{a_1}{-b_i}v_1 + \cdots + \frac{a_k}{-b_i}v_k + \frac{(-1)}{-b_i}v_{k+1} + \\ &+ \frac{b_1}{-b_i}w_1 + \cdots + \frac{b_{i-1}}{-b_i}w_{i-1} + \frac{b_{i+1}}{-b_i}w_{i+1} + \cdots + \frac{b_m}{-b_i}w_m. \end{aligned}$$

Hence

$$w_i \in \text{span}(\{v_1, \dots, v_{k+1}, w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_m\})$$

and so by Theorem 9 we have

$$\begin{aligned} &\text{span}(\{v_1, v_2, \dots, v_{k+1}, w_1, w_2, \dots, w_{i-1}, w_{i+1}, \dots, w_m\}) \\ &= \text{span}(\{v_1, v_2, \dots, v_{k+1}, w_1, w_2, \dots, w_m\}) \end{aligned}$$

and by (2) we have

$$\text{span}(\{v_1, v_2, \dots, v_{k+1}, w_1, w_2, \dots, w_m\}) = V$$

and so we are done.

QED

Theorem 18 (*Main Theorem*) *If a vector space V can be spanned by n vectors, then any set of $n + 1$ vectors in V is linearly dependent.*

proof:

Suppose for contradiction that $\text{span}(\{u_1, \dots, u_n\}) = V$ and $v_1, \dots, v_{n+1}$ are linearly independent.

Step 1. Apply the Exchange Lemma with $k = 0$ to obtain i so that $\text{span}(\{v_1, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n\}) = V$.

Step 2. Rename (relabel? reorder?) the u 's so that

$$(w_1, \dots, w_{n-1}) = (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$$

and apply the Exchange Lemma with $k = 1$ to obtain i so that

$$\text{span}(\{v_1, v_2, w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_{n-1}\}) = V.$$

Step k . Given $\{w_1, \dots, w_{n-k}\} \subseteq \{u_1, \dots, u_n\}$ such that

$$\text{span}(\{v_1, v_2, \dots, v_k, w_1, \dots, w_{n-k}\}) = V,$$

apply the Exchange Lemma to find i so that

$$\text{span}(\{v_1, v_2, \dots, v_k, v_{k+1}, w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_{n-k}\}) = V$$

Last Step: Given that $\text{span}(\{v_1, v_2, \dots, v_{n-1}, w_1\}) = V$, apply the Exchange Lemma to get that

$$\text{span}(\{v_1, v_2, \dots, v_n\}) = V.$$

But this is a contradiction, since $v_{n+1} \in \text{span}(\{v_1, v_2, \dots, v_n\})$ implies that

$$v_{n+1} = a_1 v_1 + \dots + a_n v_n$$

for some scalars a_i , but then

$$z = (-1)v_{n+1} + a_1 v_1 + \dots + a_n v_n$$

and therefore $v_1, \dots, v_{n+1}$ would be linearly dependent, a contradiction.

QED

Definition 19 The dimension of a vector space V is n , written $\dim(V) = n$ iff V has a basis of size n .

Theorem 20 Any two bases for a vector space V have the same size.

proof:

Otherwise, if $u_1, \dots, u_m$ is a basis for V and $v_1, \dots, v_n$ is another basis for V and $m < n$, then since the u 's span V it must be that the v 's are linearly dependent (by 18), contradicting that they are a basis.

QED

Theorem 21 If V is a vector space, $\dim(V) = n$, and $u_1, \dots, u_n \in V$ are linearly independent, then $u_1, \dots, u_n$ are a basis for V .

proof:

It is enough to show $\text{span}(\{u_1, \dots, u_n\}) = V$. By the main theorem (18) for any $v \in V$ we know that $u_1, u_2, \dots, u_n, v$ are linearly dependent. Hence there are scalars $a_1, a_2, \dots, a_n, a$ (at least one of which is nonzero) such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n + av = 0.$$

Since $u_1, \dots, u_n$ are linearly independent, it cannot be that $a = 0$ and so

$$v = -\frac{1}{a}(a_1u_1 + a_2u_2 + \dots + a_nu_n)$$

and so $v \in \text{span}(\{u_1, \dots, u_n\})$ and since v was an arbitrary element of V we have that $\text{span}(\{u_1, \dots, u_n\}) = V$.

QED

Theorem 22 Suppose $\dim(V) = n$ and

$$\text{span}(\{u_1, \dots, u_n\}) = V.$$

Then $u_1, \dots, u_n$ is a basis for V .

proof:

It is enough to prove that $u_1, \dots, u_n$ are linearly independent.

In a proof by contradiction
 assume the negation of what you are trying to prove
 and then reason to a contradiction.
 It follows logically
 that what you are trying to prove must be true

Suppose for contradiction that they are linearly dependent. Then there are scalars $a_1, \dots, a_n$ such that

$$a_1u_1 + \dots + a_nu_n = z$$

and for some i we have $a_i \neq 0$. But then we have that

$$u_i \in \text{span}(\{u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n\})$$

and so by Theorem 9, we have that V is spanned by $n - 1$ vectors. This would imply by the main theorem (18) that any set of n vectors is linearly dependent, contradicting the fact the dimension of V is n .

QED

Theorem 23 *Every vector space has a basis.*

proof:

This theorem is true in general but requires a more sophisticated proof for the infinite dimensional case. Here we prove it just for the case that our vector space W is a subspace of a vector space V with finite dimension. Suppose the dimension of V is n .

If $W = \{z\}$, then the dimension of W is 0 and the empty set is a basis for it. Otherwise let $v_1 \in W$ be an arbitrary vector in W not equal to the zero vector, z . If v_1 spans W , then the dimension of W is 1 and v_1 is a basis for it. Otherwise choose any $v_2 \in W$ such that $v_2 \notin \text{span}\{v_1\}$. Continue this procedure. That is given $v_1, \dots, v_k \in W$, if $v_1, \dots, v_k$ span W , then stop. Otherwise choose $v_{k+1} \in W$ arbitrary but not in the span of $v_1, \dots, v_k$. By an exercise 16 $v_1, \dots, v_k$ are linearly independent for every k . By the main theorem $k \leq n$ so this process must stop after $\leq n$ steps and when it stops we have found a basis for W .

QED

Exercise 24 *Prove:*

If $\dim(V) = n$ and $u_1, \dots, u_m \in V$ are linearly independent, then we can extend this sequence to a basis of V . That is, we can find $u_{m+1}, u_{m+2}, \dots, u_n$ so that $u_1, \dots, u_n$ is a basis for V .

Exercise 25 *If $u_1, \dots, u_n$ span a vector space V , then there exists*

$$\{v_1, \dots, v_m\} \subseteq \{u_1, \dots, u_n\}$$

such that $v_1, \dots, v_m$ is a basis for V . In other words, any spanning set contains a basis.

Theorem 26 *If W is a subspace of V and $\dim(V) = n$, then $\dim(W) = m$ for some $m \leq n$. If $m = n$ then $W = V$.*

proof:

Any basis for W is a set of m linearly independent vectors in V . But the main theorem (18) implies that $m \leq n$. Suppose $m = n$, then let $u_1, \dots, u_n$ be a basis for W . Since they are linearly independent vectors in V , by Theorem 21 they must also be a basis for V , and so

$$W = \text{span}(\{u_1, \dots, u_n\}) = V.$$

QED