

There is a total of 35 points. Put your answers on separate sheets of paper. Keep this exam, only hand in the answers. You can pick up your graded exam on Wednesday, VV403, 2-3. Your grade will be posted on the door.

1. (3 points) Find a diagonal matrix D and invertible matrix P such that $A = PDP^{-1}$ where

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

2. (6 points) Find bases for the $\text{kernel}(A)$ and the $\text{range}(A)$ where

$$A = \begin{bmatrix} 1 & 2 & -2 & -3 & 2 \\ -1 & -2 & 3 & 4 & 0 \\ 3 & 6 & -3 & -6 & -3 \end{bmatrix}$$

3. (3 points) Find an orthonormal sequence u_1, u_2, u_3 so that $\text{span}\{v_1, \dots, v_k\} = \text{span}\{u_1, \dots, u_k\}$ for $k = 1, 2, 3$.

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} \quad v_3 = \begin{bmatrix} 4 \\ -8 \\ 3 \end{bmatrix}$$

4. (3 points) Find the inverse of the matrix A :

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

5. (3 points) Find the determinant of the matrix A :

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -17 & 0 & 3 & 0 & -85 \\ 6 & -1 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 \\ 12 & 3 & 0 & 0 & -2 \end{bmatrix}$$

6. (4 points) Suppose that $A \in \mathbb{F}^{n \times m}$, $B \in \mathbb{F}^{m \times p}$, $C \in \mathbb{F}^{p \times r}$.
Prove that $(AB)C = A(BC)$.

7. (4 points) If W_1 and W_2 are subspaces of a vector space V , then define

$$W_1 \oplus W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

Prove that $W_1 \oplus W_2$ is a subspace of V .

8. (4 points) Suppose that u_1, u_2 is an orthonormal sequence in an inner product space and $c_1, c_2 \in \mathbb{R}$. Prove that $\|c_1 u_1 + c_2 u_2\| = \sqrt{c_1^2 + c_2^2}$.

9. (4 points) Suppose that $U \in \mathbb{R}^{n \times n}$ is an upper triangular matrix, U^T is its transpose, and $U^T U = U U^T$. Prove that U is a diagonal matrix.

10. (1 points) True or False: Suppose that $u_1, u_2, \dots, u_n$ is an orthonormal sequence, then $u_1, u_2, \dots, u_n$ are linearly independent.

Answers

1. $p = \begin{pmatrix} 1.000 & -2.000 \\ -0.500 & 2.000 \end{pmatrix}$ $d = \begin{pmatrix} 2.000 & 0.000 \\ 0.000 & 3.000 \end{pmatrix}$
2. basis of range: $\begin{pmatrix} 1.000 & -2.000 & 2.000 \\ -1.000 & 3.000 & 0.000 \\ 3.000 & -3.000 & -3.000 \end{pmatrix}$
 basis of kernel: $\begin{pmatrix} -2.000 & 1.000 \\ 1.000 & 0.000 \\ 0.000 & -1.000 \\ 0.000 & 1.000 \\ 0.000 & 0.000 \end{pmatrix}$
3. $\begin{matrix} & u1 & u2 & u3 \\ 0.000 & -0.600 & 0.800 \\ 1.000 & 0.000 & 0.000 \\ 0.000 & 0.800 & 0.600 \end{matrix}$
4. $\begin{pmatrix} 7.000 & -4.000 & 2.000 \\ -3.000 & 2.000 & -1.000 \\ 2.000 & -1.000 & 1.000 \end{pmatrix}$
5. -6

6.

$$\begin{aligned}
 \text{entry}_{ij}((AB)C) &= \sum_k \text{entry}_{ik}(AB) \text{entry}_{kj}(C) = \\
 &= \sum_k \sum_l \text{entry}_{il}(A) \text{entry}_{lk}(B) \text{entry}_{kj}(C) \\
 &= \sum_l \text{entry}_{il}(A) \sum_k \text{entry}_{lk}(B) \text{entry}_{kj}(C) \\
 &= \text{entry}_{ij}(A(BC))
 \end{aligned}$$

7. $c_1(w_1 + w_2) + c_2(w'_1 + w'_2) = (c_1w_1 + c_2w'_1) + (c_1w_2 + c_2w'_2)$

8. $\|c_1u_1 + c_2u_2\|^2 = \langle c_1u_1 + c_2u_2, c_1u_1 + c_2u_2 \rangle$ use linearity of inner product and $\langle u_1, u_2 \rangle = 0$ and $\langle u_1, u_1 \rangle = \langle u_2, u_2 \rangle = 1$.

9. Use that $\text{entry}_{ii}(U * U^T) = \|\text{row}_i(U)\|^2$, $\text{entry}_{ii}(U^T * U) = \|\text{col}_i(U)\|^2$ and look at minimal row i with a nonzero offdiagonal entry.

10. True