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M340 Exam 3
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There is a total of 26 points. Put your answers on separate sheets of paper. Prove everything you use. It is OK to quote any theorem that was proved in your book or in class, but you must include a proof of that theorem.

1. (5 points) Suppose that V is a vector space over the field $\mathbb{F}$, and v_j 's are vectors in V such that $v_1 \in \text{span}(\{v_2, v_3, \dots, v_m\})$. Prove that $v_1, v_2, \dots, v_m$ are linearly dependent.

2. (6 points) Suppose that the function $L : \mathbb{R}^{n \times n} \mapsto \mathbb{R}^{n \times n}$ is defined by $L(A) = A - A'$ where A' is the transpose of A . Prove that L is a linear transformation.

3. (7 points) Suppose $v \in \mathbb{C}^{n \times 1}$ and v is not the zero vector and E is the set of all $A \in \mathbb{C}^{n \times n}$ such that v is an eigenvector of A . (This means that $A \in E$ iff there exists $\lambda \in \mathbb{C}$ such that $Av = \lambda v$.) Prove that E is a subspace of $\mathbb{C}^{n \times n}$.

4. (8 points) Suppose that V is a vector space over $\mathbb{F}$, $\{u_1, u_2, \dots, u_n\}$ is a basis for V , and $u \in V$ is not the zero vector. Prove that for some i with $1 \leq i \leq n$ that

$$\{u_1, u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_n, u\}$$

is a basis for V .

1. Since $v_1 \in \text{span}(\{v_2, \dots, v_m\})$ there exists $c_2, c_3, \dots, c_m \in \mathbb{F}$ such that $v_1 = c_2v_2 + \dots + c_mv_m$. But then $\mathbf{0} = (-1)v_1 + c_2v_2 + \dots + c_mv_m$ and since the coefficient of v_1 is not zero, $v_1, v_2, \dots, v_m$ are linearly dependent.

$$\begin{aligned} 2. \quad L(A+B) &= (A+B) - (A+B)' = (A+B) - (A'+B') = A - A' + B - B' \\ &= L(A) + L(B) \\ L(cA) &= (cA) - (cA)' = cA + cA' = c(A - A') = cL(A) \end{aligned}$$

3. Since $[0]_{n \times n}v = 0v$ we have that $[0]_{n \times n} \in E$ which is the zero vector of $\mathbb{C}^{n \times n}$. If $A_0 \in E$ and $A_1 \in E$, then there exists $\lambda_0, \lambda_1 \in \mathbb{C}$ such that $A_0v = \lambda_0v$ and $A_1v = \lambda_1v$ and so

$$(A_0 + A_1)v = A_0v + A_1v = \lambda_0v + \lambda_1v = (\lambda_0 + \lambda_1)v$$

so $(A_0 + A_1) \in E$. If $c \in \mathbb{C}$ and $A \in E$, then there exists $\lambda \in \mathbb{C}$ such that $Av = \lambda v$ and

$$(cA)v = c(Av) = c(\lambda v) = (c\lambda)v$$

so $cA \in E$.

4. Since $\{u_1, u_2, \dots, u_n\}$ is a basis for V it spans V and therefore there are scalars $c_1, \dots, c_n$ such that $u = c_1u_1 + \dots + c_nu_n$. Since u is not the zero vector and $u_1, u_2, \dots, u_n$ are linearly independent some $c_i \neq 0$. For this particular i we will show that $\{u_1, u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_n, u\}$ is a basis.

Linear independence

Suppose that

$$d_1u_1 + d_2u_2 + \dots + d_{i-1}u_{i-1} + d_{i+1}u_{i+1} + \dots + d_nu_n + du = \mathbf{0}.$$

Replacing u by $c_1u_1 + \dots + c_nu_n$ and combining the coefficients we get

$$(d_1 + dc_1)u_1 + \dots + (d_{i-1} + dc_{i-1})u_{i-1} + (dc_i)u_i + (d_{i+1} + dc_{i+1})u_{i+1} + \dots$$

$$\cdots + (d_n + dc_n)u_n = \mathbf{0}$$

Since $u_1, u_2, \dots, u_n$ are linearly independent it must be that all these coefficients are zero, in particular $dc_i = 0$. But $c_i \neq 0$ so that it must be that $d = 0$. But replacing d with 0 gives

$$d_1u_1 + d_2u_2 + \cdots + d_{i-1}u_{i-1} + d_{i+1}u_{i+1} + \cdots + d_nu_n = \mathbf{0}$$

and since $u_1, u_2, \dots, u_n$ are linearly independent all the d_j 's must be zero as well. Hence $u_1, u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_n, u$ are linearly independent.

Spanning

Since $c_i \neq 0$ we can solve for u_i

$$u_i = -\frac{c_1}{c_i}u_1 + \cdots + -\frac{c_{i-1}}{c_i}u_{i-1} + -\frac{c_{i+1}}{c_i}u_{i+1} + \cdots + -\frac{c_n}{c_i}u_n + \frac{1}{c_i}u.$$

To simplify the notation write these scalars as a 's

$$u_i = a_1u_1 + \cdots + a_{i-1}u_{i-1} + a_{i+1}u_{i+1} + \cdots + a_nu_n + au.$$

Let $v \in V$ be arbitrary. Since $\{u_1, u_2, \dots, u_n\}$ is a basis for V there exists scalars $d_1, \dots, d_n$ such that $v = d_1u_1 + \cdots + d_nu_n$. Replace d_iu_i by $d_ia_1u_1 + \cdots + d_ia_{i-1}u_{i-1} + d_ia_{i+1}u_{i+1} + \cdots + d_ia_nu_n + d_iau$. and combine coefficients to get

$$\begin{aligned} v = & (d_1 + d_ia_1)u_1 + \cdots + (d_{i-1} + d_ia_{i-1})u_{i-1} + (d_{i+1} + d_ia_{i+1})u_{i+1} + \cdots \\ & \cdots + (d_n + d_ia_n)u_n + (d_ia)u \end{aligned}$$

Hence $v \in \text{span}(\{u_1, u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_n, u\})$. Since v was arbitrary $V = \text{span}(\{u_1, u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_n, u\})$.