

**There is a total of 100 points. Put your answers on separate sheets of paper. Keep this exam, hand in the answers and all work sheets.**

1. (8 points) Find all eigenvalues of

$$A = \begin{bmatrix} 3 & -4 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}?$$

2. (8 points) One eigenvalue of

$$A = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -3 & 0 \\ 1 & -1 & 4 \end{bmatrix}$$

is 4. Find an eigenvector of  $A$  (other than the zero vector) associated with the eigenvalue 4.

3. (8 points) Note that  $A = PDP^{-1}$  where:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \quad P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

What is  $A^{100}$ ?

4. (8 points) Give an example of a matrix which is not similar to a diagonal matrix and prove that it is not.

5. (12 points) Find the unique solution to the system of differential equations:

$$\begin{aligned}\frac{dx_1}{dt} &= 3x_1 - 4x_2 \\ \frac{dx_2}{dt} &= 2x_1 - 3x_2 \\ x_1(0) &= 1, \quad x_2(0) = 2\end{aligned}$$

Note:

$$\begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

6. (10 points) For  $A$  an  $m \times n$  matrix the null space of  $A$ ,  $\mathcal{N}(A)$ , is defined by

$$\mathcal{N}(A) = \{X \in \mathbf{R}^{n \times 1} : AX = Z_{m \times 1}\}.$$

Prove that  $\mathcal{N}(A)$  is a subspace of the vector space,  $\mathbf{R}^{n \times 1}$ , of all  $n \times 1$  column matrices.

7. (10 points) For each of the following sets of vectors prove they are linearly independent or prove they are linearly dependent.

(a) In the vectorspace of polynomials, the set of three vectors

$$x^2 + x, \quad x^2 - x, \quad x^3.$$

(b) In the vectorspace  $\mathbf{R}^{3 \times 1}$  the set of three vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

8. (12 points) For  $v_1, v_2, \dots, v_n$  vectors in a vector space  $V$  define
- (a)  $\text{span}\{v_1, v_2, \dots, v_n\}$ ,
  - (b)  $v_1, v_2, \dots, v_n$  are linearly independent,
  - (c)  $v_1, v_2, \dots, v_n$  are linearly dependent, and
  - (d)  $v_1, v_2, \dots, v_n$  are a basis for  $V$ .
9. (9 points) Prove that if  $v_1, v_2, \dots, v_n$  are linearly dependent, then for some  $k$  with  $1 \leq k \leq n$  we have  $v_k \in \text{span}\{v_1, v_2, \dots, v_{k-1}, v_{k+1}, \dots, v_n\}$ . Give a complete proof without referring to any other Theorem that may have been proven in class.
10. (15 points) Suppose we classify the women in a country according as to whether they live in an urban (U) or rural (R) area. Suppose each woman has just one daughter, who in turn has just one daughter, and so on. Suppose further that
- (a) for urban women, 10% of the daughters settle in rural areas and 90% settle in an urban area, and
  - (b) for rural women, 50% of the daughters settle in rural areas and 50% settle in an urban area.
1. Give the transition matrix for this Markov chain.
  2. Let  $p$  be the probability that the greatgranddaughter of an urban woman is a rural women. (woman, daughter, granddaughter, great-granddaughter) What entry of what matrix is  $p$ ? What is  $p$ ?
  3. After many generations have passed what percentage of women are urban and what percentage are rural?