

1. (20 points) Let $A \in \mathbb{F}^{m \times n}$ be a matrix which determines a 1-1 map $A : \mathbb{F}^{n \times 1} \rightarrow \mathbb{F}^{m \times 1}$. Prove that A has a left inverse.
2. (20 points) Let $A \in \mathbb{F}^{m \times n}$.
 - a. Define the column space of A , $\mathcal{C}(A)$.
 - b. Prove that $\mathcal{C}(A)$ is a subspace of $\mathbb{F}^{m \times 1}$.
3. (15 points) Let $\sigma \in S_4$ be the permutation defined by $\sigma(1) = 4, \sigma(2) = 3, \sigma(3) = 1, \sigma(4) = 2$.
 - a. What is $\text{sgn}(\sigma)$?
 - b. Write σ as a product of transpositions.
4. (10 points) a. What is the (2,3)-cofactor of A , ie $\text{cof}_{2,3}(A)$?
 b. What is $\text{entry}_{3,1}(\text{adj}(A))$?

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & -1 & 5 \\ 3 & 2 & -4 \end{bmatrix}$$

5. (10 points) Use Cramer's rule to solve the equations $AX = B$ where

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

6. (25 points) (see minimat output). Find a basis matrix for
 - a. $\mathcal{N}(\mathcal{A})$
 - b. $\mathcal{N}(\mathcal{R})$
 - c. $\mathcal{N}(\mathcal{D})$
 - d. $\mathcal{C}(\mathcal{A})$
 - e. $\mathcal{C}(\mathcal{R})$
 - f. $\mathcal{C}(\mathcal{D})$
 - g. $\mathcal{N}(\mathcal{A}^T)$
 - h. $\mathcal{C}(\mathcal{A}^T)$

```
A =
[      -3.000    -4.000     9.000
      -5.000    -7.000    15.000
      -5.000    -7.000    15.000
      -3.000    -4.000     9.000    ]
```

```
#> [R M]=rrefm(A)
```

```
R =
[      1.000     0.000    -3.000
      0.000     1.000     0.000
      0.000     0.000     0.000
      0.000     0.000     0.000    ]
```

```
M =
[      0.000     0.000     4.000    -7.000
      0.000     0.000    -3.000     5.000
      1.000     0.000    -0.000    -1.000
      0.000     1.000    -1.000    -0.000    ]
```

```
#> [DT NT]=rrefm(R'), D=DT', N=NT'
```

```
DT =
[      1.000     0.000     0.000     0.000
      0.000     1.000     0.000     0.000
      0.000     0.000     0.000     0.000    ]
```

```
NT =
[      0.000     0.000    -0.333
      0.000     1.000     0.000
      1.000     0.000     0.333    ]
```

```
D =
[      1.000     0.000     0.000
      0.000     1.000     0.000
      0.000     0.000     0.000
      0.000     0.000     0.000    ]
```

```

N =
[      0.000      0.000      1.000
      0.000      1.000      0.000
     -0.333      0.000      0.333      ]

#> invN=inv(N), invM=inv(M)

invN =
[      1.000      0.000     -3.000
      0.000      1.000      0.000
      1.000      0.000      0.000      ]

invM =
[     -3.000     -4.000      1.000      0.000
     -5.000     -7.000      0.000      1.000
     -5.000     -7.000      0.000      0.000
     -3.000     -4.000      0.000      0.000      ]

```

1. By row reduction and column reduction we can find invertible matrices $M \in \mathbb{F}^{m \times m}$ and $N \in \mathbb{F}^{n \times n}$ such that $MAN = D$ where

$$D = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Since A , M , and N are all 1-1, their composition, D , is 1-1. This means that D must have a trivial null space, or equivalently it cannot have any zero columns.

$$D = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

Noting that $D^T D = I_n$ and $A = M^{-1} D N^{-1}$, if we let $B = N D^T M$ then

$$BA = (N D^T M)(M^{-1} D N^{-1}) = N D^T (M M^{-1}) D N^{-1} = N D^T I D N^{-1} =$$

$$N D^T D N^{-1} = N I N^{-1} = N N^{-1} = I$$

so $BA = I$ and B is a left inverse of A .

2. a. $\mathcal{C}(A) = \{Y \in \mathbb{F}^{m \times 1} : \exists X \in \mathbb{F}^{n \times 1} \text{ } AX = Y\}$ or $\mathcal{C}(A) = \{AX : X \in \mathbb{F}^{n \times 1}\}$.

b. Suppose $Y_1, Y_2 \in \mathcal{C}(A)$ and $c_1, c_2 \in \mathbb{F}$. Then for some $X_1, X_2 \in \mathbb{F}^{n \times 2}$ we have $AX_1 = Y_1$ and $AX_2 = Y_2$.

$$A(c_1 X_1 + c_2 X_2) = A c_1 X_1 + A c_2 X_2 = c_1 A X_1 + c_2 A X_2 = c_1 Y_1 + c_2 Y_2$$

and so $c_1 Y_1 + c_2 Y_2 \in \mathcal{C}(A)$.

3. a. -1 b. $\langle 3, 4 \rangle \langle 2, 4 \rangle \langle 1, 3 \rangle$ or $\langle 2, 3 \rangle \langle 3, 4 \rangle \langle 1, 3 \rangle$ or $\langle 1, 4 \rangle \langle 2, 3 \rangle \langle 3, 4 \rangle$ where $\langle i, j \rangle$ is the transposition which swaps i and j .

$$4. \text{ a. } (-1)^{2+3} \det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = 4$$

$$\text{b. entry}_{3,1}(\text{adj}(A)) = \text{cof}_{1,3}(A) = (-1)^{1+3} \det \begin{bmatrix} -1 & -1 \\ 3 & 2 \end{bmatrix} = 1$$

5.

$$x_1 = \frac{\det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}}{\det \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}} = -1/3$$

$$x_2 = \frac{\det \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}}{\det \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}} = 5/3$$

6.

a. $\mathcal{N}(\mathcal{A})$ same as b or last column of N .

b. $\mathcal{N}(\mathcal{R}) \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

c. $\mathcal{N}(\mathcal{D}) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

d. $\mathcal{C}(\mathcal{A})$ First two columns of A .

e. $\mathcal{C}(\mathcal{R})$ First two columns of R .

f. $\mathcal{C}(\mathcal{D})$ First two columns of D .

g. $\mathcal{N}(\mathcal{A}^T)$ Last two columns of M^T .

h. $\mathcal{C}(\mathcal{A}^T)$ First two columns of $(N^{-1})^T$.

For g. and h. use: $MAN = D$ implies $N^T A^T M^T = D^T$

Final Exam: Tuesday, May 14, 12:25-2:25, B127 (Here).