

RULES: Do not discuss this exam with anybody except me. I will explain any definition or clarify the statement of any of these problems. My office is 403 Van Vleck, office hours MF 2:30-3:30 and W 4:30-5:30, or you can make an appointment. You may use minimat to do any calculation. The exam is due in class on Wed April 11. Exam 2 Part II will be in class on Mon April 16.

Linear differential equations.

1. A mass m_1 is hung from a spring with spring constant k_1 is attached to a rigid support. A mass m_2 is hung from a second spring with spring constant k_2 is attached to the mass m_2 . Let y_1 be the displacement from equilibrium of the mass m_1 and similarly y_2 the displacement from equilibrium of the mass m_2 . Then using Hooke's law and $F = ma$ we get that:

$$m_1 y_1'' = -k_1 y_1 + k_2 (y_2 - y_1)$$

$$m_2 y_2'' = -k_2 (y_2 - y_1).$$

Let $u_1 = y_1$, $u_2 = y_1'$, $u_3 = y_2$, and $u_4 = y_2'$. Find a matrix A such that $U' = AU$.

2. Suppose $k_1 = 21$, $k_2 = 15$, $m_1 = 7$, $m_2 = 3$ in the appropriate units. Find the general solution.

3. Suppose $y_1(0) = 1$, $y_2(0) = -1$, and both have 0 initial velocities. Find the solution. Find the displacements and velocities of the masses at time $t = 2$.

The method of least squares.

4. Prove that if A is an $n \times m$ matrix of functions of t and B is an $m \times r$ matrix of functions of t , then

$$\frac{d}{dt}(AB) = \left(\frac{d}{dt}A\right)B + A\left(\frac{d}{dt}B\right).$$

5. Prove that if U and V are $n \times 1$ vectors of functions of t , then

$$\frac{d}{dt}\langle U, V \rangle = \left\langle \frac{d}{dt}U, V \right\rangle + \left\langle U, \frac{d}{dt}V \right\rangle.$$

6. Let $A_{n \times m}, B_{m \times 1}$ be matrices of reals. Prove that

$$\frac{\partial}{\partial x_i} \|AX - B\|^2 = 2\langle AX - B, Ae_i \rangle.$$

7. Prove that if X minimizes $\|AX - B\|^2$, then $A^T(AX - B) = 0$.

8. It is good surveying practice to make more observations than are strictly necessary. For example, to determine the altitude of four points x_1, x_2, x_3, x_4 , eight observations were taken:

$$x_1 = 2.947$$

$$x_2 = 1.735$$

$$x_3 = -1.449$$

$$x_4 = 1.321$$

$$x_1 - x_2 = 1.204$$

$$x_1 - x_4 = 1.631$$

$$x_2 - x_3 = 3.186$$

$$x_3 - x_4 = -2.778$$

Note that all of these observations contain error and there do not exist any solutions to all eight equations. Find $A_{8 \times 4}$ and $B_{8 \times 1}$ such that the most likely altitudes X minimize $\|AX - B\|^2$. Find X .

Linear programming or linear optimization.

Let A be $n \times m$, B be $m \times 1$ and $C = [c_1, c_2, \dots, c_n]$ be $1 \times n$.

(*) Maximize $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ subject to the constraints:
 $AX = B$ and $X \geq 0$.

$X \geq 0$ means that $x_1 \geq 0, x_2 \geq 0, \dots$, and $x_n \geq 0$. X is called feasible iff $AX = B$ and $X \geq 0$. A solution to (*) is a vector X which is feasible and for which the value of z is as large as the value of z is for any feasible solution. X is called an extreme point iff there exists $m_1, m_2, \dots, m_k$ such that X is the unique solution of $AX = B, x_{m_1} = 0, x_{m_2} = 0, \dots$, and $x_{m_k} = 0$.

9. Give an example of (*) where there are no feasible solutions. Give an example of (*) where there are feasible solutions but no solutions. Give an example of (*) where there are there infinitely many solutions. Give an example of (*) with an extreme point which is not feasible.

10. Let X_1 and X_2 be solutions of $AX = B, x_{m_1} = 0, x_{m_2} = 0, \dots$, and $x_{m_k} = 0$. Define $L[t] = X_1 + t(X_2 - X_1)$ where t is a scalar. Show that $L[t]$ is a solution of $AX = B, x_{m_1} = 0, x_{m_2} = 0, \dots$, and $x_{m_k} = 0$ for every t .

11. Suppose in addition to above that all coordinates of X_1 other than $m_1, m_2, \dots, m_k$ are strictly positive. Show that there exists $\epsilon > 0$ such that $L[t] \geq 0$ for every t with $-\epsilon < t < \epsilon$.

12. Define $Z(t) = CL[t]$. Prove that for some scalars p and q that $Z(t) = pt + q$. Prove that Z is maximized at $t = 0$ iff $p = 0$.

13. Given the problem (*) suppose X_0 is any solution which has the maximum number of zero coordinates as any other solution. Prove that X_0 is an extreme point.

Graph theory.

Suppose that A is an $n \times n$ matrix such that for any i, j we have that $a_{i,j} = 0$ or $a_{i,j} = 1$. The matrix A determines a graph G_A on n vertices $v_1, v_2, \dots, v_n$ by the rule that v_i and v_j are connected by an edge iff $a_{i,j} = 1$. A path in a graph G is a sequence of vertices $u_1, u_2, \dots, u_k$ such that for each $i = 1, \dots, k-1$ there is an edge in G which connects u_i and u_{i+1} . A graph G is connected iff any two vertices of G can be connected by a path.

14. Give an example of a 4×4 matrix A as above whose graph is connected. Give an example of a 4×4 matrix A as above whose graph is not connected.

15. Prove that if $A^m = [c_{ij}]$, then c_{ij} is the number of paths of length m connecting v_i to v_j .

16. Prove that the graph associated to such an $n \times n$ matrix A is connected iff every entry of $I + A + A^2 + \dots + A^{n-1}$ is positive.

Markov processes.

In the Land of OZ there are three states for the weather: nice, rain, and snow. The weather follows the following rules:

- (a) There are never two nice days in a row.
- (b) When it rains or snows, half the time it the same the next day.
- (c) If the weather changes, the chances are equal for a change to each of the two other types of weather.

17. Draw a graph with vertices N, R , and S and label the edges with the appropriate probabilities.

18. Find the 3×3 transition matrix A corresponding to this graph.

19. Prove that the probability that it will rain exactly one week after it was nice is the $(1, 2)$ entry of the matrix A^7 . Find this probability.

20. Find the probability that the weather will be nice, rain, or snow on a random day in the Land of OZ.

Determinants, etc.

The trace of an $n \times n$ matrix A is defined by

$$\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

The characteristic polynomial of a matrix A is defined by $p(x) = \det(xI - A)$.

21. Prove that if A and B are similar, then $\text{trace}(A) = \text{trace}(B)$.

22. Prove that if A and B are similar, then $\det(A) = \det(B)$.

23. Prove or disprove: if two matrices A and B have the same characteristic polynomial, then they are similar.

24. Prove or disprove: if two matrices A and B are similar, then they have the same characteristic polynomial.