

There is a total of 100 points. Put your answers on separate sheets of paper. Keep this exam, hand in the answers and all work sheets.

1. (16 points) Let $A \in \mathbb{F}^{m \times n}$, $B \in \mathbb{F}^{n \times p}$, and $C \in \mathbb{F}^{p \times r}$. Prove that $A(BC) = (AB)C$.
2. (16 points) Find the inverse of the following matrix:

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

3. (16 points) Find a 4×4 elementary matrix E such that for any $4 \times n$ matrix A , EA is the matrix obtained from A by replacing the third row of A by the third row of A plus 5 times the fourth row of A . What is E^{-1} ?
4. (18 points) Find a matrix R in rref such that for some invertible matrix M , $MA = R$. Find two solutions to the equation $AX = Z$ where Z is the 3×1 zero matrix.

$$A = \begin{bmatrix} 1 & 2 & -2 & -3 & 2 \\ -1 & -2 & 3 & 4 & 0 \\ 3 & 6 & -3 & -6 & -3 \end{bmatrix}$$

5. (18 points) Let $A \in \mathbb{F}^{n \times n}$ and suppose that A is not invertible. Prove that there is a $Y \in \mathbb{F}^{n \times 1}$ for which there does not exist $X \in \mathbb{F}^{n \times 1}$ such that $AX = Y$.
6. (16 points) Let $T : V \rightarrow W$ is a mapping and $X, Y, Z \subseteq V$. Define $T(Z)$. Prove that $T(X \cap Y) \subseteq T(X) \cap T(Y)$. Give an example of T, X, Y such that $T(X \cap Y) \neq T(X) \cap T(Y)$.

1. see example 2.20 page 20.

$$2. \quad \begin{bmatrix} 7 & -4 & 2 \\ -3 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} \quad 3. \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$4. \quad R = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad X_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

5. see Thm 4.11 page 70-71.

6. Define $T(Z) = \{T(z) : z \in Z\}$, or (in English) $T(Z)$ is the set of all $T(z)$ for z an element of Z .

THM. $T(X \cap Y) \subseteq T(X) \cap T(Y)$.

proof. Let $z \in T(X \cap Y)$. Then for some $x \in X \cap Y$ we must have that $T(x) = z$. Since $x \in X \cap Y$ we have that both $x \in X$ and $x \in Y$. So that $z = T(x) \in T(X)$ and $z = T(x) \in T(Y)$ and therefore $z \in T(X) \cap T(Y)$.

Example. Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $T(x) = x^2$. Let $X = \{0, 1\}$ and $Y = \{0, -1\}$. Then $T(X \cap Y) = \{0\} \neq \{0, 1\} = T(X) \cap T(Y)$.