

You may use without proof any theorem proved in class; however when you do so, you should **state the result** you are using. You may also use any exercise, but in this case you should **state the exercise and give a full proof** of the exercise in your solution.

1. Let A be an $m \times n$ matrix, B an $n \times p$ matrix and a a scalar. Prove that

$$a(AB) = (aA)B = A(aB).$$

2. Prove that

$$\begin{aligned} &v_1, \dots, v_n \text{ are linearly dependent} \\ &\quad \text{iff} \\ &v_1 = z \text{ or } v_j \in \text{span}(\{v_1, \dots, v_{j-1}\}) \text{ for some } j \text{ with } 1 < j \leq n. \end{aligned}$$

- 3.

1. For $L : V \rightarrow W$ a function from the vector space V to the vector space W , define L is a linear transformation.
2. For $L : V \rightarrow W$ a linear transformation, define $\text{null}(L)$.
3. For $L : V \rightarrow W$ a linear transformation, define $\text{range}(L)$.
4. For $L : V \rightarrow W$ a linear transformation, define L is one-to-one.
5. For $L : V \rightarrow W$ a linear transformation, define L is onto.
6. For $\lambda \in \mathbb{C}$ and A an $n \times n$ matrix, define λ is an eigenvalue of A .
7. For $\lambda \in \mathbb{C}$ and eigenvalue of A an $n \times n$ matrix, define X is an eigenvector of A associated to λ .
8. For A an $n \times n$ matrix, define the characteristic polynomial of A .
9. For A, B two $n \times n$ matrices, define A is similar to B .
10. For A an $n \times n$ matrix, define A is diagonalizable.

4. Suppose $P_{n \times n}$ is an invertible matrix and $L : \text{MAT}_{n \times n} \rightarrow \text{MAT}_{n \times n}$ is defined by $L(A) = PAP^{-1}$ for any $A \in \text{MAT}_{n \times n}$. Prove that

1. L is a linear transformation,
2. $L(AB) = L(A)L(B)$ for every $A, B \in \text{MAT}_{n \times n}$,
3. L is one-to-one,
4. L is onto,
5. give the definition of L^{-1} . (For this particular L not the general definition of inverse)

5. Suppose that A is similar to B . Prove that A and B have the same characteristic polynomial.

Extra Credit Optional Take Home or Do-it-now Problem:

Suppose A is an $n \times n$ matrix such that $A^2 = Z$. Prove that

$$\dim(\text{range}(A)) \leq \frac{n}{2}.$$

Submit your solution via email to miller@math.wisc.edu, the earlier the better.