

You may use without proof any theorem proved in class, all of which are on the fourteen page handout; however when you do so, you should **state the result** you are using. You may also use any assigned exercise, but in this case you should **state the exercise and give a full proof** of the exercise in your solution.

1. Suppose  $V$  is a vector space,  $W$  is a subset of  $V$ , and  $v_1, v_2, \dots, v_n$  are elements of  $V$ .

Define

- (a)  $W$  is a subspace of  $V$ .
- (b)  $\text{span}(\{v_1, v_2, \dots, v_n\})$ .
- (c)  $v_1, v_2, \dots, v_n$  are linearly independent.
- (d)  $v_1, v_2, \dots, v_n$  are linearly dependent.
- (e)  $v_1, v_2, \dots, v_n$  are a basis for  $V$ .

2. A. Let  $u_1, u_2, u_3$  be elements of a vector space. Suppose

$$v_1 = u_1 - u_2$$

$$v_2 = u_2 - u_3$$

$$v_3 = u_3 - u_1$$

Prove that  $v_1, v_2, v_3$  are linearly dependent.

- B. Suppose  $u_1, u_2, u_3$  is a basis of the vector space  $V$ . Let

$$v_1 = u_1 + u_2$$

$$v_2 = u_2 + u_3$$

$$v_3 = u_1 + u_3$$

Prove that  $v_1, v_2, v_3$  is a basis of  $V$ .

3. A matrix  $A_{n \times n}$  is *Toeplitz* iff it is constant on all its ‘diagonals’. For example:

$$A = \begin{bmatrix} 0 & 6 & -4 & 0 \\ -8 & 0 & 6 & -4 \\ -2 & -8 & 0 & 6 \\ 6 & -2 & -8 & 0 \end{bmatrix}$$

So  $\text{entry}_{i,j}(A) = 0$  if  $i = j$  or  $i - j = 0$ ,  $\text{entry}_{i,j}(A) = -8$  if  $i - j = 1$ ,  $\text{entry}_{i,j}(A) = 6$  if  $i - j = -1$ , etc. More precisely an  $n \times n$  matrix  $A$  is *Toeplitz* iff for all  $i, j, i', j'$  between 1 and  $n$

$$\text{if } i - j = i' - j' \text{ then } \text{entry}_{i,j}(A) = \text{entry}_{i',j'}(A).$$

A. Prove that

$$T_n = \{A \in \text{MAT}_{n \times n} : A \text{ is Toeplitz} \}$$

is a subspace of  $\text{MAT}_{n \times n}$ . If you would prefer you may assume  $n = 4$ , i.e., show  $T_4$  is a subspace of the  $4 \times 4$  matrices.

B. Find a formula for the dimension of  $T_n$  in terms of  $n$ , i.e.,

$$\dim(T_n) = ?$$

Justify your formula by writing down a basis for  $T_3$  and  $T_4$ .

4. Let  $A$  be the following matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let  $W$  be defined as follows:

$$W = \{B \in \text{MAT}_{2,2} : AB = BA\}$$

Find a basis for  $W$  and prove that it is a basis.

5. Suppose  $A$  is any  $n \times n$  matrix. Prove there exists a polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$$

of degree  $k$  with  $1 \leq k \leq n^2$  such that

$$p(A) = a_0I + a_1A + a_2A^2 + \cdots + a_kA^k = Z$$

where  $I$  is the  $n \times n$  identity matrix and  $Z$  is the  $n \times n$  zero matrix.

Hint: What is the dimension of  $\text{MAT}_{n,n}$  and what is the size of the set  $\{0, 1, 2, \dots, n^2\}$ ?

## Answers

- 1a.  $u, v \in W$  implies  $u + v \in W$  and  $au \in W$ .
- 1b.  $\text{span}(\{u_1, \dots, u_n\}) = \{a_1u_1 + \dots + a_nu_n : a_1, \dots, a_n \text{ scalars}\}$
- 1c.  $a_1v_1 + a_2v_2 + \dots + a_nv_n = z$  implies  $a_1 = a_2 = \dots = a_n = 0$
- 1d. not linearly independent.
- 1e. linearly independent and spans  $V$ .

2A.  $v_1 + v_2 + v_3 = (u_1 - u_2) + (u_2 - u_3) + (u_3 - u_1) = z$ .

2B. We show  $v_1, v_2, v_3$  are linearly independent. Suppose

$$av_1 + bv_2 + cv_3 = z.$$

Then

$$a(u_1 + u_2) + b(u_2 + u_3) + c(u_1 + u_3) = z \text{ so } (a + c)u_1 + (a + b)u_2 + (b + c)u_3 = z.$$

Since  $u_1, u_2, u_3$  are linearly independent we have  $a + c = 0$ ,  $a + b = 0$ , and  $b + c = 0$ . Solving for  $a, b, c$  we get  $a = -c = b$  so  $a = b$ . But  $a = -b$  also so  $b = -b$  and so  $b = 0$ . Since  $a = -c = b$ , we have  $a = 0$  and  $c = 0$ . This shows  $v_1, v_2, v_3$  are linearly independent. Since any  $n$  linearly independent vectors in a vector space of dimension  $n$  is a basis, it follows that  $v_1, v_2, v_3$  is a basis.

Remark: Many students asserted without proof that

$$\text{span}(\{u_1, u_2, u_3\}) = \{(a + c)u_1 + (a + b)u_2 + (b + c)u_3 : a, b, c \in \mathbf{R}\}$$

But if  $u_1, u_2, u_3$  are linearly independent, then

$$\text{span}(\{u_1, u_2, u_3\}) \neq \{(a - c)u_1 + (b - a)u_2 + (c - a)u_3 : a, b, c \in \mathbf{R}\}$$

(This follows from the fact the  $v_1, v_2, v_3$  in part A are linearly dependent.) So something must be proved. A correct proof can be given by showing that for any  $a_1, a_2, a_3$  there exists  $a, b, c$  so that  $a + b = a_1$ ,  $b + c = a_2$ , and  $b + c = a_3$ .

3.A. Suppose  $A, B \in T_n$ . Suppose  $i - j = i' - j'$  and  $a, b$  are scalars. Then since  $A$  and  $B$  are Toeplitz,

$$\text{entry}_{i,j}(A) = \text{entry}_{i',j'}(A) \text{ and } \text{entry}_{i,j}(B) = \text{entry}_{i',j'}(B).$$

But then

$$\begin{aligned}\text{entry}_{i,j}(aA + bB) &= a\text{entry}_{i,j}(A) + b\text{entry}_{i,j}(B) = \\ a\text{entry}_{i',j'}(A) + b\text{entry}_{i',j'}(B) &= \text{entry}_{i',j'}(aA + bB)\end{aligned}$$

Hence  $(aA + bB) \in T_n$ .

$$3.B \dim(T_n) = 2n - 1.$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a basis for  $T_3$ . (Similarly for  $T_4$ .) Each basis vector is determined by a diagonal and there are  $2n - 1$  diagonals. (Count the number of entries in a  $n \times n$  matrix which are in either the first row or first column. The  $-1$  is there because the  $1, 1$  is counted twice.)

4.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

gives us

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

and therefore  $a = d$  and  $b = c$ . So

$$W = \{B \in \text{MAT}_{2,2} : AB = BA\} = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbf{R} \right\}$$

We show

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a basis for  $W$ . To see they are linearly independent, suppose

$$a \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

But then  $a = b = 0$ . To see that they span  $W$ :

$$\text{span}(\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}) = \left\{ a \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : a, b \in \mathbf{R} \right\} =$$

$$= \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbf{R} \right\} = W$$

5. The dimension of the vector space  $\text{MAT}_{n,n}$  is  $n^2$ . The **Main Theorem** proved in class is that if a vector space is spanned by  $m$  vectors, then any sequence of  $m + 1$  vectors is linearly dependent. The sequence of  $n \times n$  matrices

$$I_{n \times n}, A, A^2, A^3, \dots, A^{n^2}$$

has length  $n^2 + 1$  and is therefore linearly dependent. So there exists a nontrivial linear combination

$$a_0 I_{n \times n} + a_1 A + a_2 A^2 + \dots + a_{n^2} A^{n^2} = Z_{n \times n}$$

Let  $k$  be the largest such that  $a_k \neq 0$ , clearly  $k > 0$  because  $a_0 I \neq Z$  if  $a_0 \neq 0$ , and so

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k$$

is a polynomial with degree  $k$  ( $1 \leq k \leq n^2$ ) such that  $p(A) = Z$ .

Grading Key:

1. 4 pts each definition, 2 if partially right
2. 10 part A, 10 part B - 5 each for span, lin indep
3. 10 each part
4. 10 for a correct basis, 5 each for showing lin indep and spans
5. 2 each for  $\dim(\text{MAT}_{n,n}) = n^2$  and size of  $\{0, 1, \dots, n^2\}$  is  $n^2 + 1$ .