

1. Let  $A = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix}$ . Compute  $AB^T$ ,  $A^T B$ , and  $(AB)^T$ , or say that they have incompatible dimensions.

2. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Calculate  $A^{-1}$  if it exists, otherwise show that  $A$  is noninvertible.

3. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & x & 1 & 0 \\ 0 & 2x & x & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

For what values of  $x$ , if any, is  $A$  noninvertible?

4.

(a) State the definition:  $A$  is an invertible matrix iff ???.

(b) Let  $A$ ,  $B$ , and  $C$  be  $n \times n$  and let  $I$  be the  $n \times n$  identity matrix. Suppose  $AB = I$  and  $CA = I$ . Prove that  $B = C$  and therefore  $A$  is invertible.

5. An  $n \times n$  matrix  $B$  is called a diagonal matrix iff all entries off the diagonal are zero, i.e.,  $\text{entry}_{i,j}(B) = 0$  for any  $i \neq j$ . Let  $D$  be the  $n \times n$  diagonal matrix with  $\text{entry}_{i,i}(D) = i$  for  $i = 1, 2, \dots, n$ . So  $D$  looks like this:

$$D = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 3 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \end{bmatrix}$$

Prove that if an  $n \times n$  matrix  $A$  commutes with  $D$ , then  $A$  must be a diagonal matrix.

## Answers

1.  $(AB)^T$  undefined.

2.

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3. The determinant of  $A$  is  $2x^2 - 4x$  and this is zero iff  $x = 0$  or  $x = 2$ .

4. (a)  $A$  is invertible iff there exists a matrix  $B$  such that  $AB = I$  and  $BA = I$ .

Remarks: It is not part of the definition that  $B$  be unique. This is the point of proving part (b). You can make the definition more precise by including that  $B$  is  $n \times n$  and  $I$  is the  $n \times n$  identity matrix, if  $A$  is an  $n \times n$  matrix. This emphasises the fact that nonsquare matrices are not invertible. It cannot be the definition of invertible that the determinate is nonzero, since we have been proving theorems about invertible matrices for three weeks before the word determinant came up. You have confused the definition with a theorem about determinates.

4. (b)

proof: By the associative law of multiplication of matrices

$$(CA)B = C(AB)$$

Since  $CA = I$  we have

$$(CA)B = IB$$

Since  $I$  is the identity matrix we have

$$IB = B$$

Similarly, we have  $C(AB) = CI = C$  and so putting these together

$$B = IB = (CA)B = C(AB) = CI = C$$

and thus  $B = C$  as was to be proved. Hence  $BA = AB = I$  and therefore  $A$  is invertible. QED

Remark: You cannot assume that  $A$  is invertible, particularly since that is part of what you are asked to prove.

5. Proof: By the definition of matrix multiplication:

$$\text{entry}_{i,j}(AD) = \sum_{k=1}^n \text{entry}_{i,k}(A) \text{entry}_{k,j}(D)$$

But by definition of  $D$ ,  $\text{entry}_{k,j}(D) = 0$  if  $k \neq j$ , so

$$\sum_{k=1}^n \text{entry}_{i,k}(A) \text{entry}_{k,j}(D) = \text{entry}_{i,j}(A) \text{entry}_{j,j}(D)$$

Also,  $\text{entry}_{j,j}(D) = j$  and so  $\text{entry}_{i,j}(AD) = j \text{entry}_{i,j}(A)$ . Similarly,

$$\text{entry}_{i,j}(DA) = \sum_{k=1}^n \text{entry}_{i,k}(D) \text{entry}_{k,j}(A) = i \text{entry}_{i,j}(A)$$

Since  $AD = DA$  we have that  $\text{entry}_{i,j}(AD) = \text{entry}_{i,j}(DA)$  so

$$j \text{entry}_{i,j}(A) = i \text{entry}_{i,j}(A)$$

and if  $i \neq j$ , then this implies  $\text{entry}_{i,j}(A) = 0$  and so  $A$  is diagonal. QED

If you don't understand this proof try

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Multiply out  $AD$  and  $DA$ . What must be true if  $AD = DA$ ?

“If you can't solve the problem find a simpler version of the problem, then a solve it.”