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M331
Spring 2012

Homework problems are due in class one week from the day assigned. The date they are due proceeds them. Please do not hand in the problems early or late. But in any case do not put problems with different due dates on the same piece of paper.

It is OK to collaborate with other students on these problems. There will be no penalty. However, you should give credit where credit is due. Just mention who you were working with and/or got help from.

Mon Jan 30 **(1)** Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
Use either a Venn diagram or logical reasoning about elements.

M Jan 30 **(2)** Prove that[1] $A \setminus B = B^c \setminus A^c$ and that $A \subseteq B$ iff $B^c \subseteq A^c$.

M Jan 30 **(3)** Prove that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

M Jan 30 **(4)** Define the symmetric difference of two sets to be:

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

Prove that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

W Feb 01 **(5)** Consider the experiment of throwing four coins and noting heads H and tails T. For example HTTH means first and fourth coin H, second and third coin T.

(a) What is the size $|S|$ of S?

(b) E is the event that exactly two coins are H and two are T. List all the outcomes in E.

(c) Assume that all outcomes are equally-likely, what is $P(E)$?

W Feb 01 **(6)** $S = \{a, b, c, d\}$ and P is a probability measure on S. Suppose $a = P(\{a\})$, $b = P(\{b\})$, $c = P(\{c\})$, and $d = P(\{d\})$. Suppose b is twice a, c is three times b, and d is four times c.

What are a, b, c, and d?

What is $P(\{b, d\})$?

[1] This had an extra complement^c.

F Feb 03 **(7)** Prove $P(A \cup B \cup C) =$

$P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

F Feb 03 **(8)** A number is chosen randomly from the integers $1, 2, ..., 10000$. What is the probability that it is divisible by at least one of $2, 9$, or 13?

M Feb 06 **(9)** Suppose that $\lim_{n\to\infty} A_n = B$. This means:

$$\bigcup_{n=1}^{\infty} \bigcap_{k\geq n} A_k \;\; = \;\; B = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k$$

Prove that $\lim_{n\to\infty} P(A_n) = P(B)$.

M Feb 06 **(10)** Prove that $P(A \triangle B) = P(A) + P(B) - 2P(A \cap B)$.

M Feb 06 **(11)** You are the grader of CS101. The student are to write two programs A and B. Suppose that 50% get both right, 35% have a bug in Program A, and 45% get Program B wrong. What is the probability that a randomly chosen student got both programs wrong?

W Feb 08 **(12)** How many license plates can be made consisting of three distinct letters from A-Z followed by four distinct numerals 1-9?. For example, CZE 2936. No letter or numeral may be used more than once.

W Feb 08 **(13)** How many batting lineups can be formed from a baseball team with 15 players, three of which are pitchers, if for each line up exactly one of the pitchers plays and he bats last?

A batting line-up consists of 9 distinct players and the order in which they come up to bat: first batter, second batter, and so on, until the ninth batter.

F Feb 10 **(14)** How many ways are there to put 75 distinguishable books onto 5 distinguishable shelves? The ordering of books on each shelf counts. Assume that each possible shelving is equally likely. What is the probability that there are exactly two books on the fourth shelf?

F Feb 10 **(15)** Assume that each seven letter word with letters from A-Z is equally likely to be chosen.

(a) What is the probability that at least one letter occurs at least twice in the word? E.g., ABADEDF or PXYXXXZ

(b) What is the probability that the letter A does not occur in the word? (The probability that it does occur? The probability that occurs at least twice?)

(c) What is the probability that at least one vowel (AEIOU) occurs in the word?

M Feb 13 **(16)** (a) Write out Pascal's triangle for $0 \leq n \leq 9$.

(b) If a random number from the above triangle is chosen what is the probability that it is even?

(c) Write out $(a+b)^9$ completely expanded.

M Feb 13 **(17)** (a) What is the coefficient of x^2y^5 when $(2x-y)^7$ is completely expanded?

(b) What is the coefficient of $u^{10}v^{12}$ when $(u^2+v^3)^9$ is completely expanded?

W Feb 15 **(18)** (a) How many 8 letter words can be made by permuting the letters xxyyyzzw?

(b) Assuming each of these words is equally likely to be chosen, what is the probability that a random word begins and ends with the same letter?

(c) What is the probability that the letter y's occur consecutively?

W Feb 15 **(19)** What is the coefficient of $a^2\ b\ c^3\ d^4$ in the complete expansion of $(a+b+c+d)^{10}$?

F Feb 17 **(20)** Suppose an Urn contains 6 red, 6 blue, and 6 white balls and we simultaneously randomly choose 3 balls from the Urn.

(a) What is the probability that none of the three are red?

(b) What is the probability that at least one is red?

(c) What is the probability that all three are red?

(d) What is the probability that all three are red given that at least one is red?

(e) What is the probability that all three have different colors given that at least two colors appear?

Remark. Does it matter if instead we choose the balls sequentially without replacement? What about with replacement?

M Feb 20 **(21)** How many triples are there of integers (x, y, z) such that

$$\binom{12}{2,3,7} = \binom{12}{x}\binom{y}{z} \ ?$$

Extra credit: write a program to find them all.

M Feb 20 **(22)** In ordinary bridge each of the four players has 13 cards. There are four suits: clubs, spades, hearts, and diamonds, each of which has 13 cards.

(a) What is the probability that you are void in both clubs and diamonds?

(b) What is the probability that you are void in hearts given that your partner is void in diamonds?

M Feb 20 **(23)** Which is more probable?

(a) You and your partner have all the clubs between you.

(b) You and your partner are both void in spades.

W Feb 22 **(24)** Monty Hall's Lets make a deal. Suppose the following:

(1) Contestant randomly chooses one of three doors. Behind one of the doors is a fabulous **P**rize and the other two doors a **G**oat.

(2) After opening one of the doors not chosen and displaying a goat, Monty Hall does the following:

If the contestant has chosen **P**, then with probability p Monty Hall offers to switch doors. If the contestant has chosen **G**, then with probability q Monty Hall offers to switch doors.

(3) If the contestant is allowed to switch doors, then with probability r he or she takes the switch offer.

(a) Draw a tree representing all outcomes of this "Deal".

(b) As a function of p, q, and r what is the probability that the Contestant gets the **P**rize? (i.e., winning)

(c) Show that for any fixed values of p and q that either $r = 0$ or $r = 1$ will maximize the probability that the Contestant wins.

(d) For what values of p and q is it true that the probability of winning is the same for all values of r?

(e) Show that for any s with $0 \leq s \leq 1$ there are values p, q such that if the contestant always takes the switch when offered ($r = 1$), then the probability of the Contestant winning is s.

F Feb 24 **(25)** How many 8 letter words can be made by permuting the letters xxyyyzzw and have the property that no two y's occur next to each other?

F Feb 24 **(26)** A basket contains 5 White balls and 6 Green balls.

1. Choose one ball from the basket at random.

2. If it is White replace it in the basket with 2 Green. If it is Green replace it with 3 White. In either case do not put the chosen ball back in the basket.

3. Randomly choose a second ball from the basket.

(a) What is the probability that the first ball was White given that the second ball chosen was Green? i.e., $P(W_1|G_2)$.

(b) What is $P(G_1|W_2)$?

M Feb 27 **(27)** Given S, H, W_1, and W_2 in some probability space, let $p_1 = P(S|W_1)$ and $p_2 = P(S|W_2)$. S and H are complimentary events.[2] Consider the following three statements:

(1) $S \cap W_1$, $S \cap W_2$ are independent and $H \cap W_1$, $H \cap W_2$ are independent.

(2)

$$P(S|W_1 \cap W_2) = \frac{p_1 p_2}{p_1 p_2 + (1 - p_1)(1 - p_2)}$$

(3)

$$\frac{P(H \cap W_1 \cap W_2)}{P(S \cap W_1 \cap W_2)} = \frac{P(H \cap W_1)P(H \cap W_2)}{P(S \cap W_1)P(S \cap W_2)}$$

Assume all six probabilities in statement (3) are positive.
Prove that (1) $\rightarrow$ (2).

M Feb 27 **(28)** Prove that (2) $\rightarrow$ (3) and (3) $\rightarrow$ (2).

M Feb 27 **(29)** Give an example where (3) is true (and therefore (2)) but (1) is false.

W Feb 29 **(30)** Here is a way[3] to use randomness to encourage people to tell the truth when answering an embarrassing survey question. The survey has one question:

[2] These stand for Spam and Ham, see Wikipedia on Bayesian spam filtering.

[3] This is from Scientific American June 2011 page 30.

Q. Have you ever "X-ed"?

Where "X-ed" stands for some embarrassing X-rated sexual activity for which people are likely to lie and say No even if that have in fact X-ed.

Randomized Procedure: Call up a random person on the phone and ask them to flip a coin but not tell you of the outcome. If the coin comes up H they are to say yes to the question whatever the true answer is. If the coin comes up T they are to give the true answer.

The idea is that the person is more likely to give the true answer since the researcher does not know if the person answered yes because the coin flip was H or because it was T and the person truly X-ed. After asking 1000 people and receiving 550 yes answers the researcher can guess that about 500 got H and answered yes and 500 got T and 50 answered yes and so about $\frac{50}{500}$ or 10% have X-ed.

Suppose you are told that the probability a random person has X-ed is p. In terms of p what is the probability that the randomly chosen person has X-ed given that he answered yes when he was called and the randomized procedure above was followed? Check your answer for p equal 10%.

Extra credit. What happens if instead the person flips the coin twice and always answer yes except if TT occurs and then the person answers truthfully?

W Feb 29 **(31)** Suppose A_1, A_2, A_3 are independent. Show that A_1^c, A_2, A_3 are independent. Remark: By symmetry it follows that any of the eight sequences B_1, B_2, B_3 is independent where each B_k is either A_k or A_k^c.

Recall that A_1, A_2, A_3 are independent iff the following are all true:

1. $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$
2. $P(A_1 \cap A_2) = P(A_1)P(A_2)$
3. $P(A_1 \cap A_3) = P(A_1)P(A_3)$
4. $P(A_2 \cap A_3) = P(A_2)P(A_3)$

W Feb 29 **(32)** Give an example of A_1, A_2, A_3 where 1,2,3 are all true but 4 is false.

W Feb 29 **(33)** Give an example of A_1, A_2, A_3 where 2,3,4 are all true but 1 is false.

F Mar 2 **(34)** Suppose that A, B are independent and A, C are independent. Prove that $A, B \cup C$ are independent iff $A, B \cap C$ are independent.

F Mar 2 **(35)** This is a multiple choice problem.
Q. If you randomly chose one of the following answers, what is the probability that you chose correctly?

(a) 25%
(b) 50%
(c) 50%
(d) 100%

M Mar 5 **(36)** Three distinguishable dice are thrown. The random variable X is the number of different numbers that appear, e.g., $X(\langle 5, 3, 5 \rangle) = 2$. Find $E(X)$.

W Mar 7 **(37)** Suppose a basket contains 10 red, 20 white, and 30 blue balls. Thirty are chosen randomly at the same time. Let R be the number of red balls chosen, W the number of white balls chosen, and B the number of blue balls chosen. Find $E(R + 2W - B)$.

F Mar 9 **(38)** Find an example of three ball players A, B, and C and two seasons such that in terms of their batting average:

A beats B each season and B beats C each season, but C beats B and B beats A when the season stats are combined.

(World Series of Baseball) Suppose that a coin has probability p of falling Heads. Repeatedly toss it until at least four H appear or four T, i.e., best out of seven games is the winner. Let X be the number of tosses this takes, i.e., $4 \leq X \leq 7$

F Mar 9 **(39)** In terms of p what is the probability distribution of X?

F Mar 9 **(40)** For $p = \frac{1}{2}$ in problem (39) find the expectation $\mu = E(X)$, variation $\nu =$var(X), and the standard deviation σ of X.

M Mar 12 **(41)** (Russian Roulette) The Master of Ceremonies (MC) puts one bullet into a six shooter and randomly rotates the chamber. There are six players who one-by-one take the pistol put it to there head and squeeze the trigger. X is the player that shoots himself. (A less bloody description would be: throw a die. X is the number showing on the die) Find the expected value, variation, and standard deviation of X.

M Mar 12 **(42)** (Russian Roulette Variation) The Master of Ceremonies (MC) puts one bullet into a six shooter and randomly rotates the chamber. He gives it to player 1 who squeezes the trigger. If he shoots himself the game is over. Otherwise the MC randomly rotates the chamber again and gives the pistol to player 2. Y is the player who is shot. If all players survive one round then $Y = 0$. (A less bloody description would be: throw a die 6 times. Let Y be the first time a one is thrown, if any is, otherwise $Y = 0$ if there are no ones.) Find the expected value, variation, and standard deviation of Y.

M Mar 12 **(43)** Suppose X and Y are random variables on the same probability space

$$P(X = 2) = \frac{1}{3} \qquad P(Y = 1) = \frac{1}{4}$$
$$P(X = 3) = \frac{1}{3} \qquad P(Y = 2) = \frac{1}{4}$$
$$P(X = 6) = \frac{1}{3} \qquad P(Y = 3) = \frac{1}{2}$$

(a) Compute $E(X)$, $E(Y)$, and $E(X)E(Y)$.

(b) Suppose X and Y are independent and compute $E(XY)$.

(c) Suppose $P(X = a \text{ and } Y = b) = \frac{1}{9}$ for all a, b except[4]

$$P(X = 6 \text{ and } Y = 1) = \frac{1}{4} - \frac{2}{9}$$
$$P(X = 6 \text{ and } Y = 2) = \frac{1}{4} - \frac{2}{9}$$
$$P(X = 6 \text{ and } Y = 3) = \frac{1}{2} - \frac{2}{9}$$

[4] I made a mistake in class - they can't all be $\frac{1}{9}$.

then compute $E(XY)$.

W Mar 14 **(44)** Does $E(XY) = E(X)E(Y)$ imply that X, Y are independent? No. Suppose $A, B \subseteq \Omega$ have probability ϵ and are disjoint and a, b are real numbers. Define

$$X(\omega) = \begin{cases} a & \text{if } \omega \in A \\ 1 & \text{if } \omega \in B \\ 0 & \text{otherwise} \end{cases} \qquad Y(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ b & \text{if } \omega \in B \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E(X)$, $E(Y)$, $E(X)E(Y)$, and $E(XY)$.

(b) Show that there is an ϵ with $0 < \epsilon < \frac{1}{2}$ and a, b different from 1, so that X, Y are not independent but $E(XY) = E(X)E(Y)$.

W Mar 14 **(45)** (a) Give an example of a random variable X such that $E(X) = 2$ and $E(X^2) = 3$ or show that this is impossible. (b) Give an example of a random variable X such that $E(X) = 1$ and $E(X^2) = 3$ or show that this is impossible.

W Mar 14 **(46)** Let X and Y be random variables (which may not be independent) but for which $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$. Prove that $E(XY) = E(X)E(Y)$.

F Mar 16 **(47)** Suppose a basketball player has a probability of making a 1-point shot (free throw) of $\frac{1}{2}$, a 2-point shot of $\frac{1}{3}$, and a 3-point shot of $\frac{1}{5}$. During a game he makes 10 1-point attempts, 21 2-point attempts, and 5 3-point attempts. Let X be the total number of points he scores. Assuming all shot attempts are pairwise independent Bernoulli trials, find the expected value, variance, and standard deviation of X.

M Mar 19 **(48)** Let $X_1, X_2, \dots, X_n$ be pairwise independent Bernoulli trials with probability of success $p = \frac{1}{3}$. Using Chebyshev find the smallest n such that the probability is at least 99% that $X_1 + X_2 + \cdots + X_n$ is $\frac{n}{3}$ with error no worse than plus or minus one percent of $\frac{n}{3}$.

M Mar 19 **(49)** Suppose $X \geq 0$ and $P(X < 3) < \frac{1}{5}$. If $\mu = E(X)$ what is the smallest possible value μ can have?

Remark: If X takes on negative values also, could μ be smaller?

M Mar 19 **(50)** Suppose σ is the standard deviation of X and μ its expectation. Prove:

(a) $P(\mu - 2\sigma < X < \mu + 2\sigma) \geq .75$
(b) $P(\mu - 3\sigma < X < \mu + 3\sigma) \geq .88$

F Mar 23 **(51)** (Drunkard's walk) Define a sequence

$$x_0 = 0 \qquad x_{k+1} = \begin{cases} 0 & \text{if } k \geq 1 \text{ and } x_k = 0 \text{ otherwise:} \\ x_k + 1 & \text{with probability } \frac{1}{2} \\ x_k - 1 & \text{with probability } \frac{1}{2} \end{cases}$$

The drunk staggers back and forth randomly but if he ever returns to the origin he stops, i.e., from that point on x_k is constantly zero. Use the Monte Carlo method to estimate the probability that $x_{10} = 0$. Use any programming language you want and any large number of trials, say a million. With a reliability of 95% how close is your estimate to the exact value? What is your estimate of the probability that $x_{50} = 0$ or the probability that $x_{100} = 0$?

F Mar 23 **(52)** Suppose that $u^2 \leq v$. Construct a random variable X with $E(X) = u$ and $E(X^2) = v$.

F Mar 23 **(53)** Suppose that the score of a student on an exam is modeled by a random variable X. Given that the expected value of X is 68 points, find an upper bound on the probability that the student scores 78 or more.

F Mar 23 **(54)** Suppose that it also known that the standard deviation of X is 6 points. Find a better upper bound on the probability that the student scores 78 or more.

F Mar 23 **(55)** Suppose that there are n students in the class and their scores on the exam are modeled by pairwise independent random variables with the same mean 68 and standard deviation 6 as in the previous two problems. Estimate the smallest value of n such that the probability is at least 95% that the average score of the class on the exam is within 5 points of the mean 68.

M Mar 26 **(56)** In the Las Vegas game of KENO (which is similar to a lotterie) a player chooses (marks) from 1 to 15 numbers out of 80 and makes a bet. The casino randomly chooses 20 out these 80. It pays out an amount that depends on the number of "catches" or "winning spots" the player's card has. For more information on KENO see "How to play Keno".

	catch	Bally: bet 10 pays out	Barbary: bet 1 pays out
Mark 5	3	10	1
	4	200	25
	5	5500	600

Let X be a random variable which is the amount Bally casino will payout on a 10 dollar bet where the player has marked 5 spots on his card. Find the expected value of X. Similarly let Y be the amount that Barbary casino will pay out on a 1 dollar bet. Find the expected value of X and the expected value of Y.

Let Z_k be the number of catches for a card with k spots marked out of a possible 80. Find the expected value and variance of Z_k for $k = 5, 6, 7$.

M Mar 26 **(57)** Beeno is like Keno but uses a binomial distribution. Instead of choosing exactly 20 out of 80 the casino chooses each number independently with probability $\frac{1}{4}$. On average this will be 20. Repeat problem (56) for the analogous variables, X', Y', Z'_k.

W Mar 28 **(58)** Suppose a woman has probability b of having a boy and g of having a girl. Assume $0 < b < 1$ and $g = 1 - b$. She decides to keep having children until she has a child of each sex. Let X be the number of children she has in the end. Find the expected value, variance, and standard deviation (μ, ν, σ) of X in terms of b and g.

W Mar 28 **(59)** For each of $b = \frac{1}{2}$ and $b = \frac{1}{3}$ in problem (58) compute the exact values of μ, ν, and σ.

F Mar 30 **(60)** Given $0 < p < 1$ and $p + q = 1$, suppose that for every $n = 1, 2, \ldots$ that $P(X = n) = q^{n-1}p$. Find

$$E\left(\frac{1}{X!}\right)$$

Extra credit: find $E(X^3)$ or $E(\frac{1}{X})$.

F Mar 30 **(61)** Give an example of a discrete random variable $X \geq 0$ such that $E(X)$ is finite but $E(X^2) = \infty$.

W Apr 11 **(62)** Let $0 < p < 1$ and $p + q = 1$. Fix n and let

$$r_k = \binom{n}{k} p^k q^{n-k}$$

(a) Prove for all k that $r_k < r_{k+1}$ iff $k < np - q$.

(b) Prove for all k that $r_k \geq r_{k+1}$ iff $k \geq np - q$.

(c) Define $M_n = \max\{r_k : k = 0, 1, 2, \ldots, n\}$. Show that $M_n = r_{k_0}$ where k_0 is the least k such that $np - q \leq k$.

(d) For $n = 10$ and $p = 1/3$ find M_n.

(e) For $n = 11$ and $p = 2/3$ find M_n.

Remark. Is k_0 always the closest integer to np? If $np - q = k$, then is $r_k = r_{k+1}$?

F Apr 13 **(63)** Suppose you have 6 red balls and 6 black balls in a basket. Consider the following algorithm:

1. Choose a ball randomly from the basket.
2. If it is a red ball, put it back in the basket.
3. If it is a black ball, remove it from the basket.
4. If there are any black balls still in the basket, go to step 1.
5. Stop.

Find the expected value of the number of times that step 1 is executed.

F Apr 13 **(64)** Tom, Dick, and Harry flip a silver dollar repeatedly in the order:

T, D, H, T, D, H, T, D, H, T, ...

until finally one of them gets a HEAD and wins the dollar. For each of the players find the expected value of the amount they win. Assume that the probability of HEADs is p with $0 < p < 1$ and TAILs is q with $p + q = 1$. What are these three values when $p = \frac{1}{2}$? What happens as $p \to 0$?

M Apr 16 **(65)** Auto accidents occur at the corner of University and Charter at the Poisson rate of three per month. What is the probability that at least two accidents will occur during the month of June?

M Apr 16 **(66)** (Continuation) What is the probability that no accidents occur during exactly three of the next six months?

M Apr 16 **(67)** Let X_P be Poisson with parameter λ. Show that

$$E(X_P^3) = \lambda \, E((X_P + 1)^2)$$

and use it to compute $E(X_P^3)$.

W Apr 18 **(68)** Suppose Bob starts with \$10 and Mike starts with \$30. They repeatedly bet \$1 until one of them goes broke. Let p be the probability that Bob wins each individual bet. For each of $p = .5,\ .49$, and $.45$, find the probability that Bob goes broke before Mike does.

W Apr 18 **(69)** Repeat problem (68) but instead assume that each individual bet is for \$5 instead of \$1.

F Apr 20 **(70)** Six red and six blue marbles are randomly put into two urns, A and B. Urn A has two marbles and Urn B has ten marbles. One marble from A and one marble from B are chosen randomly and switched. Draw a Markov transition graph and give the transition matrix for this problem.

F Apr 20 **(71)** Let

$$A_1 = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 0 & -1 \end{bmatrix}$$

For all nine possible pairs A_i and A_j compute A_iA_j or say that it is undefined. For how many of the 27 triples is $A_iA_jA_k$ defined?

W Apr 25 **(72)** The Raisin-Bran Game. On the back of box of raisin bran cereal appears the following game. There are six squares numbered

(1) Start
(2) OK
(3) OK
(4) OK
(5) GoBack to (2)
(6) Finish

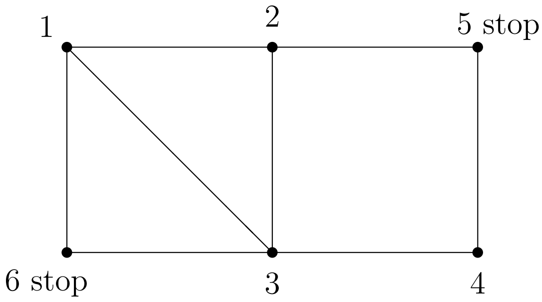
The game starts at (1) and ends when you arrive at (6). For each move you throw a fair coin, if it lands Heads you advance from (i) to (i+1), if it lands Tails you advance from (i) to (i+2). If you land on (5) you immediately return to (2). Landing on (5) and returning to (2) counts as one move, not two.

Find the expected number of moves in the Raisin-Bran game.

Remark. It is easy to find an on-line matrix inverter calculator. For example, bluebit looks easy to use:

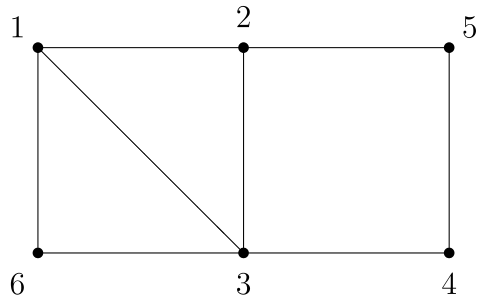
http://www.bluebit.gr/matrix-calculator/

F Apr 27 **(73)** Take a random walk along this graph:



At any of the nodes 1,2,3,4 the chain is equally likely to follow any of the out-going edges. At the nodes marked stop the Markov process terminates. For each of the nodes 1-4 calculate the probability that a random walk starting there will eventually arrive at 5 and the probability it will eventually arrive at 6.

F Apr 27 **(74)** Same graph but no stop states:

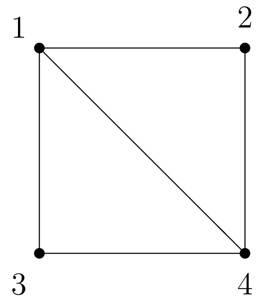


Prove that this Markov chain is regular.

M Apr 30 **(75)** Prove that the random walk on an n-cycle is regular iff n is odd.

W May 2 **(76)** In a long line of children you happen to notice that about 30% of the time a girl is followed by a boy and about 80% of the time a boy is followed by another boy. If you randomly choose a child from the line, what is the probability that its a girl?

W May 2 **(77)** For a random walk on the graph:



find the steady state probability vector.

W May 2 **(78)** (from the Snell book) Suppose that P is the transition matrix for an ergodic Markov chain. Let

$$Q = \frac{1}{2}I + \frac{1}{2}P$$

(a) Prove that Q is regular.
(b) Prove that if $\vec{s}\,Q = \vec{s}$, then $\vec{s}\,P = \vec{s}$.

F May 4 **(79)** Calculate the Lebesgue measure of the set

$$\bigcup_{n=1}^{\infty} \left(n! + \frac{1}{3^n},\ \ n! + \frac{1}{2^n}\right]$$

M May 7 **(80)** Fix K a large integer and $s = (s_0, \ldots, s_{K-1}) \in 2^{<\omega}$ a finite sequence of zeros and ones. Define

$$D_n = \{x \in 2^\omega \ :\ x_{n+i} = s_i \text{ for all } i = 0, \ldots, K-1\}$$

Show that

$$\mu(\bigcap_{n=0}^{\infty} \bigcup_{k>n} D_k) = 1$$

M May 7 **(81)** Define

$$E_n = \{x \in 2^\omega \ : \ x_{n+i} = 0 \text{ for all } i = 0, \ldots, n-1\}$$

Show that

$$\mu\left(\bigcap_{n=0}^{\infty} \bigcup_{k>n} E_k\right) = 0$$

W May 9 **(82)** Suppose in Feller's infinite URN game, at each of the steps $n = 1, 2, \ldots$ you add enough to have $n^2 + 1$ balls in the URN and then you randomly choose one of the balls in the URN and discard it. Prove that the probability that ball 1 is never discarded is positive.

Remark. With probability 1 infinitely many of the balls are never discarded. Note that you must add $(n^2 + 1) - (n - 1)^2 = 2n$ balls at step n. This result remains true if you add $2n$, n, or even $\frac{1}{2}n$ balls at step n.

W May 9 **(83)** Suppose x, y, z are three random reals in $[0, 1]$ chosen independently and uniformly. What is the probability that there is a triangle with sides of length x, y, and z?

Practice Problems - Do not hand in.

Practice Problem **(84)** Prove that $A_1, \ldots, A_n$ are independent if and only if for any sequence $B_1, \ldots, B_n$ where each B_k is either A_k or A_k^c we have that

$$P(B_1 \cap B_2 \cap \cdots \cap B_n) = P(B_1)\, P(B_2) \cdots P(B_n)$$

Practice Problem **(85)** Suppose we have two red, two white, two blue balls in a basket.

Choose 2, if they have the same color you win.

Choose 2 more (without replacing first two), if they have the same color you win.

Finally if the two left in the basket are the same color, you win.

Would you take an even money bet? I.e., if you win you get a dollar, if you lose you pay a dollar. What is the probability that you win?

Practice Problem **(86)** There are two fair die, a red one and a blue one.

A is the event that the red die is a 1 or 6.

B is the event that the total of the two die is 7,

C is the event that the red die is a 1 or the blue die is a 6.

For each of the following determine if is true or false and explain why.

1. A and B are independent events.
2. A and C are independent events.
3. B and C are independent events.
4. A and $B \cup C$ are independent events.
5. $A \cap C$ and $B \cup C$ are independent events.

Practice Problem **(87)** You have 8 cards, 4 aces and 4 kings,

a. Choose three at random simultaneously. What is the probability that one is a king given that 2 are aces?

b. Choose three in order 1,2,3 without replacement. What is the probability that card 3 is a king given that cards 1,2 are aces.

c. Choose three at random and suppose at least 2 are aces. Randomly permute them and sequentially show then 1,2,3. What is probability the card 3 is a king given 1,2 are aces?

d. Choose three at random simultaneously, What is the probability all are kings given that not all are aces?

Practice Problem **(88)** Prove or disprove;

Suppose A,B,C are independent, must $A \cup B$ and $B \cup C$ be independent?

Practice Problem **(89)** A box contains 3 cards:

one card has both sides white

one card has both sides black

one card has a white side and a black side

A card is randomly drawn out of the box and placed on a table

Given that the side facing up is black what is the probability that the other side of the card is black?

Practice Problem **(90)** In problem (30) change the randomization procedure:

If first flip is H, then flip again and say yes if H and no if T. If first flip is T then answer truthfully.

Practice Problem **(91)** For any regular Markov chain transition matrix P show that there must be k such that for all $m \geq k$ all entries of P^m are positive.

Practice Problem **(92)** Assume X_k's are pairwise independent Bernoulli variables with parameter p. Use Chebyshev's inequality and the Borel-Cantelli Lemma to show that for any $\epsilon > 0$ the probability that there are infinitely many n with

$$\left| \frac{X_1 + \ldots X_{n^2}}{n^2} - p \right| \geq \epsilon$$

is zero.

Practice Problem **(93)** For independent uniformly distributed random variables x and y in the unit interval $[0, 1]$ find the expected value of the distance between x and y.

Practice Problem **(94)** For a random point P in the unit disk find the expected value of r, the distance from P to the origin. Hint: you can differentiate the cummalative distribution function to find the density function for r.

Practice Problem **(95)** For a pair of independent random reals in the unit interval what is the expected value of xy? Why should it be $\frac{1}{2} \cdot \frac{1}{2}$?

Practice Problem **(96)** In Laplace's famous example of "what is the probability that the sun rises tomorrow" he shows the following:

Randomly choose a probability p in the unit interval and toss a coin with probability p of being a Head. Then the probability it comes out Heads n-times in a row is

$$\int_0^1 p^n \, dp = \frac{1}{n+1}$$

Show that for any k that

$$\int_0^1 \binom{n}{k} p^k (1-p)^{n-k} \, dp = \frac{1}{n+1}$$

Hint: use integration by parts. Can you give a probabilistic argument?

Practice Problem **(97)** In Bertrand's random chord paradox[5] suppose we pick a random chord on the unit disk using the following procedures:

(4) Randomly choose two points inside the unit disk and take the chord containing them.

(5) Randomly choose one point inside the unit disk and one point on the circle and take the chord containing them.

(6) Randomly choose a length in $[0, 2]$ and then a chord of that length.

(7) Randomly choose an area $0 \leq A \leq \frac{\pi}{2}$ and choose a chord cutting off that area from the unit disk.

What is the probability that the chord chosen has length greater than one of the sides of the inscribed equilateral triangle?

Can you come up with an eight method?

Practice Problem **(98)** Suppose you choose a random real number x in $[0, 1]$. What is the probability that x is a rational number?

[5] wikipedia.org : Bertrand's paradox