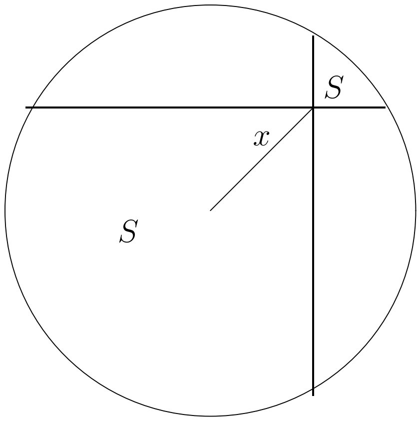

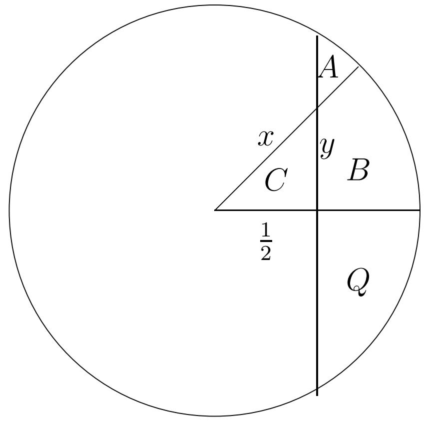

97. Suppose we pick a random chord on the unit disk by randomly choosing two points inside the unit disk and take the chord containing them. What is the probability that the chord chosen has length greater than one of the sides of the inscribed equilateral triangle?

Answer:

Let the first point be chosen inside the disk and suppose its distance from the center is x. Draw the two chords of lenght $\sqrt{3}$ going through the point. If the second point is between these two chords then the random chord is longer than $\sqrt{3}$. See figure on left. Denote the area between the two chords as $S(x)$

Note that if $0 < x < \frac{1}{2}$ then $S(x) = \pi$, i.e., the area of the unit disk. The probability that the first point is inside the disk of radius $\frac{1}{2}$ is $\frac{1}{4}$. The probability we are looking for is:

$$\frac{1}{4} + \int_{\frac{1}{2}}^{1} \frac{S(x)}{\pi} 2x \, dx$$

This is because in the Riemann sum

$$\sum_{k=1}^{n} \frac{S(x_k)}{\pi} 2x_k \triangle x$$

$2x_k \triangle x$ approximates the probability the first point lands in the annulus of width $\triangle x$ at radius x_k and $\frac{S(x_k)}{\pi}$ approximates the probability the second point lies inside the S-region for x's close to x_k.

To calculate $S(x)$ we use the figure on the right.

$A + B = Q$
$\frac{1}{4} + y^2 = x^x$
$C = \frac{y}{4}$
$C + B = \frac{\theta}{2}$ where θ is the angle at the center.
$\frac{1}{2} = x\cos(\theta) \quad$ so $\theta = \text{arcsec}(2x)$
Area T of inscribed equillateral triangle is $\frac{3}{4}\sqrt{3}$.
$6Q + T = \pi$
$S(x) = \pi - 2(2Q - 2A)$
Solving these gives

$$S(x) = \pi - 2\ \text{arcsec}(2x) + \sqrt{x^2 - \frac{1}{4}}$$

The integral $\int x\ \text{arcsec}(x)\ dx$ can be done by parts.
We end up with

$$\frac{1}{4} + \int_{\frac{1}{2}}^{1} \frac{S(x)}{\pi} 2x\ dx = \frac{1}{12} + \frac{3\sqrt{3}}{2\pi} \quad \sim \quad .91$$