Math 331 | 5/9/12 | Jake Darlington

5-9-82

Suppose in Feller's Infinite URN game, at each of the steps $n = 1, 2, 3, \ldots$ you add enough to have n^2+1 balls in the URN and then you randomly choose one of the balls in the urn and discard it. Prove that the probability that ball 1 is never discarded is positive.

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83 A

$\Rightarrow$ At each turn, there are n^2+1 balls in urn. If ball 1 is never to be discarded, then only $(n^2+1)-1 = n^2$ balls are eligible to be drawn and discarded each turn.

$\Rightarrow$ Prob(Ball 1 never discarded) $= \underset{\uparrow \atop n=1}{\frac{1}{2}} \times \underset{\uparrow \atop n=2}{\frac{4}{5}} \times \underset{\uparrow \atop n=3}{\frac{9}{10}} \times \cdots \cong \prod_{n=1}^{\infty}\left(\frac{n^2}{n^2+1}\right)$

$\Rightarrow$ Need to show $\sim \prod_{n=1}^{\infty}\left(n^2/n^2+1\right)$ is positive (i.e. $\lim_{N\to\infty}\prod_{n=1}^{N}\left(\frac{n^2}{n^2+1}\right) \neq 0$)

$\Rightarrow E = \bigcap_{n=1}^{N} E_n \qquad \mu(E_n) = \frac{n^2}{n^2+1} \quad \Rightarrow \quad \mu(E) = \lim_{N\to\infty}\left(\bigcap_{n=1}^{N} E_n\right) = \lim_{N\to\infty}\left(\prod_{n=1}^{N}\frac{n^2}{n^2+1}\right)$

$\Rightarrow \ln\left(\prod_{n=1}^{N}\frac{n^2}{n^2+1}\right) = \sum_{n=1}^{N}\ln\left(\frac{n^2}{n^2+1}\right) \Rightarrow \lim_{N\to\infty}\sum_{n=1}^{N}\ln\left(\frac{\cancel{n^2}}{\cancel{n^2}(1+\frac{1}{n^2})}\right) \Rightarrow \lim_{N\to\infty}\sum_{n=1}^{N}\ln\left(\frac{1}{1+\frac{1}{n^2}}\right) \longrightarrow \ln(1)$ as $N\to\infty$

$\Rightarrow \ln(1) = 0 \Rightarrow$ hence as $\sum_{n=1} \ln\left(\frac{n^2}{n^2+1}\right)$ does not go to $-\infty$ as $N\to\infty$, then $\lim_{N\to\infty}\prod_{n=1}^{\infty}\left(\frac{n^2}{n^2+1}\right)$ does not go to zero as $N\to\infty$.

$\Rightarrow$ Hence $\lim_{N\to\infty}\prod_{n=1}^{N}\left(\frac{n^2}{n^2+1}\right) \neq 0 \Rightarrow$ hence probability that ball 1 is never discarded is positive.

Check: using Wolfram-Alpha, $\prod_{n=1}^{\infty}\frac{n^2}{n^2+1} \cong 0.272029$

$\Rightarrow$ probability that ball 1 is never discarded is positive.

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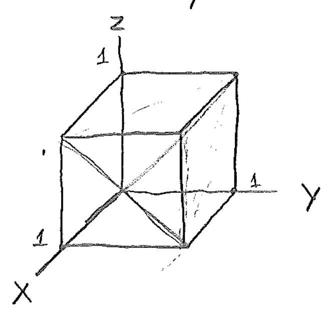
5-9-83 Suppose x, y, z are three random reals in $[0,1]$ chosen independently and uniformly. What is the probability that there is a triangle with sides x, y, + z?

$\Rightarrow$ For $x, y, + z$ to make a triangle: $x \le y+z$, $y \le x+z$, $z \le x+y$

$\Rightarrow$ Need to find volume inside unit cube bounded by the three inequalities above in order to calculate the probability. $\rightarrow$ Volume (R)

$\Rightarrow$ $\text{Prob} = \dfrac{\text{Volume}(R)}{\text{Volume}(\text{cube})} = \dfrac{1-(\text{Vol}(R))^c}{1}$ where $(\text{Vol}(R))^c =$ volume of corners cut off by the compliments of the inequalities $x > y+z$, $z > x+y$, $y > x+z$

$\Rightarrow$ find these volumes by triple integrals



Volume for $x > y+z$

$$\Rightarrow \int_0^1\int_0^x\int_0^{x-y} dz\,dy\,dx = \int_0^1\int_0^x (x-y)\,dy\,dx = \int_0^1 \left[xy - \frac{y^2}{2}\Big|_0^x\right]dx$$

$$= \int_0^1 x^2 - \frac{x^2}{2}\,dx = \int_0^1 \frac{x^2}{2}\,dx = \frac{x^3}{6}\Big|_0^1 = \frac{1}{6}$$

Volume for $y > x+z$

$$\Rightarrow \int_0^1\int_0^y\int_0^{y-x} dz\,dx\,dy = \int_0^1\int_0^y [y-x]\,dx\,dy = \int_0^1\left[yx - \frac{x^2}{2}\Big|_0^y\right]dy = \int_0^1 \frac{y^2}{2}\,dy = \frac{y^3}{6}\Big|_0^1 = \frac{1}{6}$$

Volume for $z > x+y$

$$\Rightarrow \int_0^1\int_0^{1-x}\int_{x+y}^{1} dz\,dy\,dx = \int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx = \int_0^1\left[y - xy - \frac{y^2}{2}\Big|_0^{1-x}\right]dx$$

$$= \int_0^1\left[(1-x) - x(1-x) - \frac{(1-x)^2}{2}\right]dx = \int_0^1 1 - x - x + x^2 - \frac{1}{2} + \frac{2x}{2} - \frac{x^2}{2}\,dx = \int_0^1 \frac{1}{2} - x + \frac{1}{2}x^2\,dx$$

$$= \left[\frac{1}{2}x - \frac{x^2}{2} + \frac{1}{6}x^3\Big|_0^1\right] = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}$$

$$\Rightarrow (\text{Vol}(R))^c = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} \qquad \Rightarrow \text{Prob} = \frac{1 - 1/2}{1} = \boxed{\frac{1}{2}}$$

$\Rightarrow$ $\boxed{\text{Probability } x, y, \text{ and } z \text{ are the sides of a triangle} = \frac{1}{2}}$