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M331
5/7/2012
Homework

80. Fix K a large integer and $s = (s_0, s_1, \ldots, s_{K-1}) \in 2^{<\omega}$ a finite sequence of zeros and ones. Define

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$$D_n = \{x \in 2^\omega : x_{n+i} = s_i \ \ \forall\ i = 0, 1, \ldots, K-1\}$$

Show that

$$\mu\left(\bigcap_{n=0}^{\infty} \bigcup_{k \geq n} D_k\right) = 1$$

Let $E_n = D_{n \cdot K}$, then $\bigcup\limits_{i \geq m}^{\infty} E_i \subseteq \bigcup\limits_{i \geq m}^{\infty} D_i$ for any $m \in \mathbb{N}^0$, $m \geq 0$

This implies that:

$$\bigcap_{n=0}^{\infty} \bigcup_{m \geq n} E_m \subseteq \bigcap_{n=0}^{\infty} \bigcup_{m \geq n} D_m$$

Which means:

$$\mu\left(\bigcap_{n=0}^{\infty} \bigcup_{m \geq n} E_m\right) \leq \mu\left(\bigcap_{n=0}^{\infty} \bigcup_{m \geq n} D_m\right)$$ (since $\mu(x)$ is a probability distribution function on x).

Given that E_n's are independent, and

$\mu(E_n) = \frac{1}{2^K}$ $\forall\ n$, then $\sum\limits_{n=0}^{\infty} \mu(E_n) = \sum\limits_{n=0}^{\infty} \frac{1}{2^K} = \infty$

Thus, given independence and $\sum\limits_{n=0}^{\infty} \mu(E_n) = \infty$, $\mu\left(\bigcap\limits_{n=0}^{\infty} \bigcup\limits_{m \geq n} E_m\right) = 1$

Given $0 \leq \mu\left(\bigcap\limits_{n=0}^{\infty} \bigcup\limits_{m \geq n} D_m\right) \leq 1$ by definition, and

$\mu\left(\bigcap\limits_{n=0}^{\infty} \bigcup\limits_{m \geq n} D_m\right) \geq \mu\left(\bigcap\limits_{n=0}^{\infty} \bigcup\limits_{m \geq n} E_m\right) = 1$,

$\mu\left(\bigcap\limits_{n=0}^{\infty} \bigcup\limits_{m \geq n} D_m\right) = 1$

81. Define $E_n = \{x \in 2^\omega : x_{n+i} = 0 \ \forall \ i = 0, 1, \ldots, n-1\}$
Show that
$$\mu\left(\bigcap_{n=0}^{\infty} \bigcup_{k \geq n} E_k\right) = 0$$

For all E_n, $\mu(E_n) = \frac{1}{2^n}$ since there must be n-zeroes after x_n of $x \in 2^\omega$.

Hence, $\sum_{n=0}^{\infty} \mu(E_n) = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-(1/2)} = 2.$

By theorem, if $\sum_{n=0}^{\infty} \mu(E_n) < \infty$, $\mu\left(\bigcap_{n=0}^{\infty} \bigcup_{k \geq n} E_k\right) = 0$

Thus, given $\sum_{n=0}^{\infty} \mu(E_n) = 2 < \infty$,

$$\mu\left(\bigcap_{n=0}^{\infty} \bigcup_{k \geq n} E_k\right) = 0$$