Math 331 | 4/11/12 | Jake Darlington

4-11-62 Let $0<p<1$ and $p+q=1$. Fix n and let

$$r_K = \binom{n}{K} p^K q^{n-K}$$

62A extra

a) Prove for all K that $r_K < r_{K+1}$ iff $K < np-q$.

Left to right: $r_K < r_{K+1}$

$\Rightarrow \binom{n}{K} p^K q^{n-K} < \binom{n}{K+1} p^{K+1} q^{(n-(K+1))}$

$\Rightarrow \dfrac{\binom{n}{K}}{\binom{n}{K+1}} < \dfrac{\cancel{p^{K+1} q^{(n-K-1)}}}{\cancel{p^K q^{(n-K)}}}$

$\Rightarrow \dfrac{\cancel{\frac{n!}{(n-K)!\,K!}}}{\cancel{\frac{n!}{(K+1)!\,(n-(K+1))!}}} < \dfrac{p}{q}$

$\Rightarrow \dfrac{K+1}{n-K} < \dfrac{p}{q}$

$\Rightarrow qK+q < pn-pK$

$qK+pK < pn-q$

$\Rightarrow K(p+q) < np-q$ but $p+q=1$

$\Rightarrow \underline{K < np-q}$

Right to left: $K < np-q$

$\Rightarrow$ multiply by $1=p+q$ $\Rightarrow$ $K(p+q) < np-q$

$Kq+q < np-pK$

$q(K+1) < p(n-K)$

$\Rightarrow \dfrac{(K+1)}{(n-K)} < \dfrac{p}{q}$ next multiply by more forms of 1

$\Rightarrow \dfrac{(K+1)!\,(n-(K+1))!}{K!\,(n-K)!} < \dfrac{p^{K+1}}{p^K}\dfrac{q^{(n-(K+1))}}{q^{(n-K)}}$

$\Rightarrow \dfrac{\frac{n!}{K!(n-K)!}}{\frac{n!}{(K+1)!(n-(K+1))!}} < \dfrac{p^{K+1} q^{(n-(K+1))}}{p^K q^{(n-K)}}$

$\Rightarrow \binom{n}{K} p^K q^{(n-K)} < \binom{n}{K+1} p^{K+1} q^{(n-(K+1))}$

$\Rightarrow \underline{r_K < r_{K+1}}$

$\boxed{\text{Hence, } r_K < r_{K+1} \text{ iff } K < np-q, \text{ for all } K}$

b) Prove for all K that $r_K \geq r_{K+1}$ iff $K \geq np-q$

Left to right: $r_K \geq r_{K+1}$

$\Rightarrow \binom{n}{K} p^K q^{n-K} \geq \binom{n}{K+1} p^{(K+1)} q^{(n-K-1)}$

$\Rightarrow \dfrac{\frac{n!}{K!(n-K)!}}{\frac{n!}{(K+1)!(n-K-1)!}} \geq \dfrac{p^{(K+1)} q^{(n-K-1)}}{p^K q^{n-K}}$

$\Rightarrow \dfrac{K+1}{n-K} \geq \dfrac{p}{q} \Rightarrow qK+q \geq pn-pK$

$\Rightarrow K(p+q) \geq np-q$

$\Rightarrow \underline{K \geq np-q}$

Right to left: $K \geq np-q$

$\Rightarrow K(p+q) \geq np-q \Rightarrow Kq+q \geq -Kp+np$

$\Rightarrow q(K+1) \geq p(n-K)$

$\Rightarrow \dfrac{K+1}{n-K} \geq \dfrac{p}{q} \Rightarrow \dfrac{(K+1)!\,(n-K-1)!}{K!\,(n-K)!} \geq \dfrac{p}{q}$

$\Rightarrow \dfrac{\frac{n!}{K!(n-K)!}}{\frac{n!}{(K+1)!(n-(K+1))!}} \geq \dfrac{p^{K+1} q^{n-K-1}}{p^K q^{n-K}}$

$\Rightarrow \binom{n}{K} p^K q^{n-K} \geq \binom{n}{K+1} p^{K+1} q^{(n-(K+1))}$

$\Rightarrow \underline{r_K \geq r_{K+1}}$

$\Rightarrow \boxed{\text{Hence, } r_K \geq r_{K+1} \text{ iff } K \geq np-q, \text{ for all } K}$

4-11-62 cont $\quad M_n = \max\{r_k : k = 0, 1, 2, \ldots, n\}$

c) Show that $M_n = r_{k_0}$ where k_0 is the least k such that $k \geq np-q$

$\Rightarrow M_n = \max r_k \rightarrow$ hence $r_{k_0} \geq r_{k_0+1} \Rightarrow \frac{r_{k_0}}{r_{k_0+1}} \geq 1$

$$\frac{r_{k_0}}{r_{k_0+1}} = \frac{\binom{n}{k_0} p^{k_0} q^{n-k_0}}{\binom{n}{k_0+1} p^{k_0+1} q^{n-k_0-1}} = \frac{\frac{n!}{k_0!(n-k_0)!} p^{k_0} q^{n-k_0}}{\frac{n!}{(k_0+1)!(n-k_0-1)!} p^{k_0+1} q^{n-k_0-1}} = \frac{(k_0+1)q}{(n-k_0)p} \geq 1$$

$\Rightarrow qk_0 + q \geq np - pk_0 \Rightarrow (q+p)k_0 \geq np - q \Rightarrow \underline{k_0 \geq np - q}$

$\boxed{\text{Hence, } M_n = r_{k_0} \text{ where } k_0 \text{ is the least } k \text{ such that } k \geq np-q}$

d) $n = 10$, $p = \frac{1}{3}$, $M_n = ?$

$\Rightarrow np - q = 10\left(\frac{1}{3}\right) - \left(\frac{2}{3}\right) = 2.67 \qquad k \geq 2.67 \Rightarrow \underline{k = 3}$

$M_n = \binom{10}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^7 = \frac{10!}{7!\,3!}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^7 = 0.26012 \Rightarrow$ $\boxed{n = 10,\ p = 1/3 \Rightarrow M_n = 0.26012}$

e) $n = 11$, $p = \frac{2}{3}$, $M_n = ?$

$\Rightarrow np - q = 11\left(\frac{2}{3}\right) - \frac{1}{3} = 7 \qquad k \geq 7 \Rightarrow k = 7$

$M_n = \binom{11}{7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)^4 = \frac{11!}{7!\,4!}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)^4 = 0.23845 \Rightarrow$ $\boxed{n = 11,\ p = 2/3 \Rightarrow M_n = 0.23845}$

<u>Remark</u> 1) Is k_0 always the closest integer to np?

$\Rightarrow$ <u>No</u>, k_0 is not always the closest integer to np.

example: Let $n = 22$, $p = \frac{1}{6}$, $q = \frac{5}{6}$. Then $np = 22\left(\frac{1}{6}\right) = 3.667$

<u>Closest integer to np is 4.</u> But $k_0 \overset{\text{least } k}{\geq} np - q = 3.67 - \left(\frac{5}{6}\right) = 2.83$

$\Rightarrow \underline{k_0 = 3}$. $\boxed{\text{Hence } k_0 \text{ is not always the closest integer to } np}$

2) If $np - q = k$, is $r_k = r_{k+1}$

$r_k = r_{k+1} \Rightarrow \frac{n!}{(n-k)!\,k!} p^k q^{n-k} = \frac{n!}{(n-k-1)!\,(k+1)!} p^{k+1} q^{n-k-1} \Rightarrow \frac{k+1}{n-k} = \frac{p}{q} \Rightarrow kq + q = np - kp$

$\Rightarrow k(p+q) = np - q \Rightarrow \underline{k = np - q} \Rightarrow$ $\boxed{\text{Hence, if } k = np - q, \text{ then } r_k = r_{k+1}}$