Math 331

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Due Feb 27

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#27

Suppose $S\cap W_1$, $S\cap W_2$ are independent and $H\cap W_1$, $H\cap W_2$ are independent.

Then $P(S\cap W_1 \cap S\cap W_2) = P(S\cap W_1)P(S\cap W_2) = p_1 \cdot p_2 P(W_1)\cdot P(W_2)$

and $P(H\cap W_1 \cap H\cap W_2) = P(H\cap W_1)P(H\cap W_2) = (1-p_1)(1-p_2)P(W_1)P(W_2)$

Note that $P(W_1\cap W_2) = P(W_1\cap W_2\cap S) + P(W_1\cap W_2\cap H)$

$= p_1p_2 P(W_1)P(W_2) + (1-p_1)(1-p_2)P(W_1)P(W_2)$

because $S^c = H$.

Now,

$$P(S\mid W_1\cap W_2) = \frac{P(S\cap W_1\cap W_2)}{P(W_1\cap W_2)}$$

$$= \frac{p_1p_2 P(W_1)\cdot P(W_2)}{p_1p_2 P(W_1)\cdot P(W_2) + (1-p_1)(1-p_2)P(W_1)P(W_2)}$$

$$= \frac{p_1p_2}{p_1p_2 + (1-p_1)(1-p_2)}.$$

#28 (2) ⇒ (3)

$S^c = H$.

Note that $P(W_1) = P(S \cap W_1) + P(S^c \cap W_1)$ implies $1 = p_1 + \frac{P(S^c \cap W_1)}{P(W_1)}$ and hence $(1-p_1) = P(H \mid W_1)$

Similarly $(1-p_2) = P(H \mid W_2)$.

In addition, $P(W_1 \cap W_2) = P(S \cap W_1 \cap W_2) + P(H \cap W_1 \cap W_2)$, and

$$P(S \cap W_1 \cap W_2) = P(S \mid W_1 \cap W_2) \cdot P(W_1 \cap W_2)$$

Now,

$$\frac{P(H \cap W_1 \cap W_2)}{P(S \cap W_1 \cap W_2)} = \frac{P(W_1 \cap W_2) - P(S \cap W_1 \cap W_2)}{P(S \cap W_1 \cap W_2)}$$

$$= \frac{\frac{P(S \cap W_1 \cap W_2)}{P(S \mid W_1 \cap W_2)} - P(S \cap W_1 \cap W_2)}{P(S \cap W_1 \cap W_2)}$$

$$= \frac{1}{P(S \mid W_1 \cap W_2)} - 1$$

$$= \frac{p_1 p_2 + (1-p_1)(1-p_2)}{p_1 p_2} - 1 \qquad \text{(by (2))}.$$

$$= 1 + \frac{(1-p_1)(1-p_2)}{p_1 p_2} - 1$$

$$= \frac{P(H \cap W_1)\, P(H \cap W_2)}{P(S \cap W_1)\, P(S \cap W_2)}$$

(3) ⇒ (2)

Assume (3).

By above, I have $\frac{P(H \cap W_1) P(H \cap W_2)}{P(S \cap W_1) P(S \cap W_2)} = \frac{(1-p_1)(1-p_2)}{p_1 p_2} = \frac{P(H \cap W_1 \cap W_2)}{P(S \cap W_1 \cap W_2)} = \frac{1}{P(S \mid W_1 \cap W_2)} - 1$.

Then $P(S \mid W_1 \cap W_2) = \left(\frac{(1-p_1)(1-p_2)}{p_1 p_2} + 1 \right)^{-1}$

$$= \frac{p_1 p_2}{(1-p_1)(1-p_2) + p_1 p_2}$$

#29.

Let $P(S) = P(W_1) = P(W_2) = \frac{1}{2}$ and let S, W_1, W_2 be independent.

Then $$\frac{P(H \cap W_1 \cap W_2)}{P(S \cap W_1 \cap W_2)} = \frac{\frac{1}{8}}{\frac{1}{8}} = 1 = \frac{\frac{1}{4} \cdot \frac{1}{4}}{\frac{1}{4} \cdot \frac{1}{4}} = \frac{P(H \cap W_1)\, P(H \cap W_2)}{P(S \cap W_1)\, P(S \cap W_2)}$$

So (3) holds. But,

$$P(S \cap W_1 \cap S \cap W_2) = P(S \cap W_1 \cap W_2) = \frac{1}{8} \neq \frac{1}{16} = P(S \cap W_1)\, P(S \cap W_2)$$

So (1) fails.

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2-27-27 Given S, H, W_1, W_2 in some probability space, let $p_1 = P(S|W_1)$, $p_2 = P(S|W_2)$
$S = H^c$

Consider following statements:

(1) $(S \cap W_1)$, $(S \cap W_2)$ are independent and $(H \cap W_1)$, $(H \cap W_2)$ are independent.

(2) $$P(S|W_1 \cap W_2) = \frac{p_1 p_2}{p_1 p_2 + (1-p_1)(1-p_2)}$$

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(3) $$\frac{P(H \cap W_1 \cap W_2)}{P(S \cap W_1 \cap W_2)} = \frac{P(H \cap W_1) P(H \cap W_2)}{P(S \cap W_1) P(S \cap W_2)}$$ $\rightarrow$ all these 6 probabilities > 0

Prove $(1) \rightarrow (2)$

$$\Rightarrow P(S|W_1 \cap W_2) = \frac{P(S \cap (W_1 \cap W_2))}{P(W_1 \cap W_2)} = \frac{P((S \cap W_1) \cap (S \cap W_2))}{P(S \cap (W_1 \cap W_2)) + P(S^c \cap (W_1 \cap W_2))}$$

use (1) from above $\rightarrow$ independence $\Rightarrow P(A \cap B) = P(A)P(B)$

$$\Rightarrow \frac{P(S \cap W_1) P(S \cap W_2)}{P(S \cap W_1) P(S \cap W_2) + P(S^c \cap W_1) P(S^c \cap W_2)}$$

use $S^c = H$, $P(A \cap B) = P(A|B)P(B)$

$$= \frac{P(S|W_1) \cancel{P(W_1)} P(S|W_2) \cancel{P(W_2)}}{P(S|W_1) \cancel{P(W_1)} P(S|W_2) \cancel{P(W_2)} + P(H|W_1) \cancel{P(W_1)} P(H|W_2) \cancel{P(W_2)}}$$

use $P(S|W_1) + P(H|W_1) = 1$ as $S = H^c$

$$= \frac{P(S|W_1) P(S|W_2)}{P(S|W_1) P(S|W_2) + (1 - P(S|W_1))(1 - P(S|W_2))} \Rightarrow \frac{p_1 p_2}{p_1 p_2 + (1-p_1)(1-p_2)} \therefore$$

$\boxed{\Rightarrow \text{Hence, } (1) \rightarrow (2)}$

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2-27-28 Prove (2) → (3), and (3) → (2)

Worked with Zach Knoeck

ok

(2) → (3):

$$P(S|W_1 \wedge W_2) = \frac{P(S|W_1)P(S|W_2)}{P(S|W_1)P(S|W_2) + (1-P(S|W_1))(1-P(S|W_2))}$$

$$\Rightarrow \frac{P(S\wedge(W_1\wedge W_2))}{P(W_1\wedge W_2)} = \frac{\frac{P(S\wedge W_1)}{\cancel{P(W_1)}}\cdot\frac{P(S\wedge W_2)}{\cancel{P(W_2)}}}{\frac{P(S\wedge W_1)}{\cancel{P(W_1)}}\cdot\frac{P(S\wedge W_2)}{\cancel{P(W_2)}} + \frac{P(H\wedge W_1)}{\cancel{P(W_1)}}\cdot\frac{P(H\wedge W_2)}{\cancel{P(W_2)}}}$$

$$\Rightarrow \frac{P(W_1\wedge W_2)}{P(S\wedge W_1\wedge W_2)} = \frac{P(S\wedge W_1)P(S\wedge W_2) + P(H\wedge W_1)P(H\wedge W_2)}{P(S\wedge W_1)P(S\wedge W_2)}$$

Use the following: $P(W_1\wedge W_2) = P(S\wedge(W_1\wedge W_2)) + \underbrace{P(S^c\wedge(W_1\wedge W_2))}_{= P(H\wedge(W_1\wedge W_2))}$

$$\Rightarrow \frac{P(S\wedge(W_1\wedge W_2)) + P(H\wedge(W_1\wedge W_2))}{P(S\wedge W_1\wedge W_2)} = \frac{P(S\wedge W_1)P(S\wedge W_2) + P(H\wedge W_1)P(H\wedge W_2)}{P(S\wedge W_1)P(S\wedge W_2)}$$

$$\Rightarrow \frac{P(S\wedge W_1\wedge W_2)}{P(S\wedge W_1\wedge W_2)} + \frac{P(H\wedge W_1\wedge W_2)}{P(S\wedge W_1\wedge W_2)} = \frac{P(S\wedge W_1)P(S\wedge W_2)}{P(S\wedge W_1)P(S\wedge W_2)} + \frac{P(H\wedge W_1)P(H\wedge W_2)}{P(S\wedge W_1)P(S\wedge W_2)}$$

$$\Rightarrow \cancel{1} + \frac{P(H\wedge W_1\wedge W_2)}{P(S\wedge W_1\wedge W_2)} = \cancel{1} + \frac{P(H\wedge W_1)P(H\wedge W_2)}{P(S\wedge W_1)P(S\wedge W_2)}$$

$\Rightarrow$ therefore: $\boxed{\frac{P(H\wedge W_1\wedge W_2)}{P(S\wedge W_1\wedge W_2)} = \frac{P(H\wedge W_1)P(H\wedge W_2)}{P(S\wedge W_1)P(S\wedge W_2)}}$ hence (2) → (3)

(3) → (2) Use reverse steps:

$$\Rightarrow \frac{P(H\wedge W_1\wedge W_2)}{P(S\wedge W_1\wedge W_2)} = \frac{P(H\wedge W_1)P(H\wedge W_2)}{P(S\wedge W_1)P(S\wedge W_2)} \Rightarrow \frac{P(S\wedge W_1\wedge W_2) + P(H\wedge W_1\wedge W_2)}{P(S\wedge W_1\wedge W_2)} = \frac{P(S\wedge W_1)P(S\wedge W_2) + P(H\wedge W_1)P(H\wedge W_2)}{P(S\wedge W_1)P(S\wedge W_2)}$$

Flip

$$\Rightarrow \frac{P(S\wedge W_1\wedge W_2)}{P(W_1\wedge W_2)} = \frac{P(S\wedge W_1)P(S\wedge W_2)}{P(S\wedge W_1)P(S\wedge W_2) + P(H\wedge W_1)P(H\wedge W_2)} \Leftarrow \text{multiply by } \left(\frac{P(W_1)P(W_2)}{P(W_1)P(W_2)}\right)$$

$$\Rightarrow P(S|W_1\wedge W_2) = \frac{P(S|W_1)P(S|W_2)}{P(S|W_1)P(S|W_2) + P(H|W_1)P(H|W_2)} = \frac{P(S|W_1)P(S|W_2)}{P(S|W_1)P(S|W_2) + (1-P(S|W_1))(1-P(S|W_2))}$$

$\Rightarrow$ therefore: $\boxed{P(S|W_1\wedge W_2) = \frac{p_1 p_2}{p_1 p_2 + (1-p_1)(1-p_2)}}$ hence (3) → (2)

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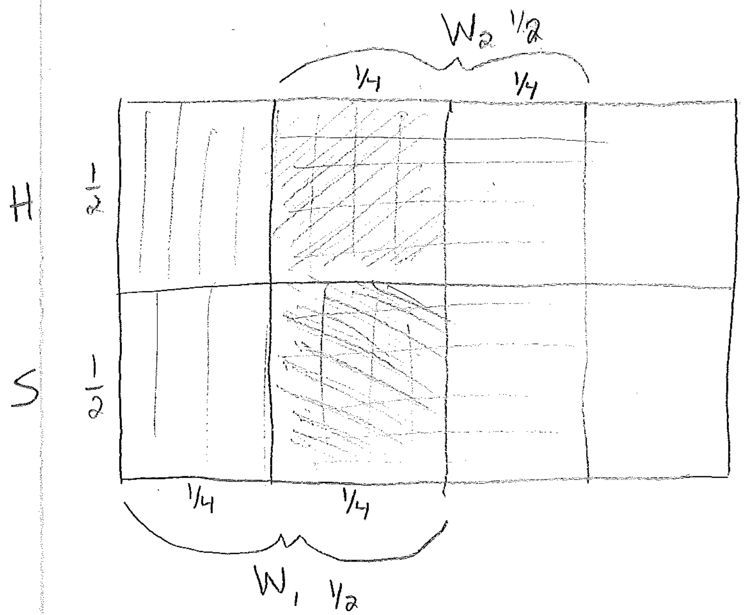
2-27-29 Give example showing (3) $\not\Rightarrow$ (1)

Worked w/ Zach Knoeck

Show example where (3) is true but (1) is false



From (3): $\dfrac{P(H \cap W_1 \cap W_2)}{P(S \cap W_1 \cap W_2)} = \dfrac{P(H \cap W_1)P(H \cap W_2)}{P(S \cap W_1)P(S \cap W_2)} \Rightarrow \dfrac{\frac{1}{8}}{\frac{1}{8}} = \dfrac{\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)}{\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)}$

$\Rightarrow \dfrac{\frac{1}{8}}{\frac{1}{8}} = \dfrac{\frac{1}{16}}{\frac{1}{16}}$ ✓ (3) holds

From (1) $\rightarrow$ $(S \cap W_1), (S \cap W_2)$ ind. and $(H \cap W_1), (H \cap W_2)$ ind.

$P((S \cap W_1) \cap (S \cap W_2)) \stackrel{?}{=} P(S \cap W_1)P(S \cap W_2) \Rightarrow \frac{1}{8} \neq \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$ ✗ (1) fails

$P((H \cap W_1) \cap (H \cap W_2)) \stackrel{?}{=} P(H \cap W_1)P(H \cap W_2) \Rightarrow \frac{1}{8} \neq \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$ ✗ (1) fails

Hence, the above example shows a situation where (3) is true but (1) is false