Math 331 | 2/20/12 | Jake Darlington

2-20-21 How many triples are there of integers (x,y,z) such that
$$\binom{12}{2,3,7} = \binom{12}{x}\binom{y}{z}?$$

Step 1: The first triples can be found by breaking up 12 into a group of 2, a group of 3, and a group of 7.

For example: $\binom{12}{2,3,7} = \binom{12}{2}\binom{10}{7} \Rightarrow$ hence the triple is $(2,10,7)$

12 triples can be found in this step

$$\Rightarrow \binom{12}{2,3,7} = \overset{1}{\binom{12}{2}\binom{10}{7}} = \overset{2}{\binom{12}{2}\binom{10}{3}} = \overset{3}{\binom{12}{3}\binom{9}{7}} = \overset{4}{\binom{12}{3}\binom{9}{2}} = \overset{5}{\binom{12}{5}\binom{5}{3}} = \overset{6}{\binom{12}{5}\binom{5}{2}} = \overset{7}{\binom{12}{7}\binom{5}{3}}$$
$$= \overset{8}{\binom{12}{7}\binom{5}{2}} = \overset{9}{\binom{12}{9}\binom{9}{7}} = \overset{10}{\binom{12}{9}\binom{9}{2}} = \overset{11}{\binom{12}{10}\binom{10}{3}} = \overset{12}{\binom{12}{10}\binom{10}{7}}$$

Step 2: The remaining triples must be found by using the equality of $\binom{12}{2,3,7} = \binom{12}{x}\binom{y}{z}$

x must be ≤ 12, while $z \leq y$.

$$\binom{12}{2,3,7} = \frac{12!}{2!\,3!\,7!} = 7920$$

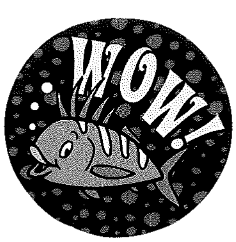
If $x = 1$, for example, then $\binom{12}{2,3,7} = \binom{12}{1}\binom{y}{z} \Rightarrow 7920 = 12\binom{y}{z} \Rightarrow 660 = \binom{y}{z}$

Then, by using Excel, I searched through combinations of $\binom{y}{z}$, constrained by $z \leq y$, so that $\binom{y}{z} = 660$. In this case, $\binom{y}{z} = \binom{660}{659} = \binom{660}{1} = 660$

Hence, there are two triples with $x=1 \Rightarrow$ $(1, 660, 659)$, and $(1, 660, 1)$.

This process can be repeated for values of $x \leq 12$.

The full chart is shown on the next page.



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$\binom{12}{2,3,7} = 7920$

X	$\binom{12}{X}$	$\binom{y}{z} = \frac{7920}{\binom{12}{X}}$	Multinomial $\binom{y}{z} =$	Triple (x,y,z)	# of triples
0	1	7920	$\binom{7920}{7919}, \binom{7920}{1}$	(0, 7920, 7919), (0, 7920, 1)	2
1	12	660	$\binom{660}{659}, \binom{660}{1}$	(1, 660, 659), (1, 660, 1)	2
2	66	120	$\binom{16}{14}, \binom{16}{2}, \binom{120}{119}, \binom{119}{1}$	(2, 16, 14), (2, 16, 2), (2, 120, 119), (2, 119, 1)	4
3	220	36	$\binom{36}{35}, \binom{36}{1}$	(3, 36, 35), (3, 36, 1)	2
4	495	16	$\binom{16}{15}, \binom{16}{1}$	(4, 16, 15), (4, 16, 1)	2
5	792	10	$\binom{10}{9}, \binom{10}{1}$	(5, 10, 9), (5, 10, 1)	2
6	924	8.57	none	none	0
7	792	10	$\binom{10}{9}, \binom{10}{1}$	(7, 10, 9), (7, 10, 1)	2
8	495	16	$\binom{16}{15}, \binom{16}{1}$	(8, 16, 15), (8, 16, 1)	2
9	220	36	$\binom{36}{35}, \binom{36}{1}$	(9, 36, 35), (9, 36, 1)	2
10	66	120	$\binom{16}{14}, \binom{16}{2}, \binom{120}{119}, \binom{120}{1}$	(10, 16, 14), (10, 16, 2), (10, 120, 119), (10, 120, 1)	4
11	12	660	$\binom{660}{659}, \binom{660}{1}$	(11, 660, 659), (11, 660, 1)	2
12	1	7920	$\binom{7920}{7919}, \binom{7920}{1}$	(12, 7920, 7919), (12, 7920, 1)	2

Sum = 28

Hence, by running X from 0 to 12, 28 triples were found in this step.

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Step 3: Combine Steps 1 + 2:

I found 40 triples of integers (x,y,z) such that $\binom{12}{2,3,7} = \binom{12}{x}\binom{y}{z}$

They are listed here:

1) (0, 7920, 7919)
2) (0, 7920, 1)
3) (1, 660, 659)
4) (1, 660, 1)
5) (2, 10, 7)
6) (2, 10, 3)
7) (2, 16, 14)
8) (2, 16, 2)
9) (2, 120, 119)
10) (2, 120, 1)
11) (3, 9, 7)
12) (3, 9, 2)
13) (3, 36, 35)
14) (3, 36, 1)
15) (4, 16, 15)
16) (4, 16, 1)
17) (5, 5, 3)
18) (5, 5, 2)
19) (5, 10, 9)
20) (5, 10, 1)
21) (7, 5, 3)
22) (7, 5, 2)
23) (7, 10, 9)
24) (7, 10, 1)
25) (8, 16, 15)
26) (8, 16, 1)
27) (9, 9, 7)
28) (9, 9, 2)
29) (9, 36, 35)
30) (9, 36, 1)
31) (10, 10, 3)
32) (10, 10, 7)
33) (10, 16, 14)
34) (10, 16, 2)
35) (10, 120, 119)
36) (10, 120, 1)
37) (11, 660, 659)
38) (11, 660, 1)
39) (12, 7920, 7919)
40) (12, 7920, 1)