

No calculators.
No notes.
Show all work.
Circle your answer.
Hand in to your TA.

Name_____

Circle the time of your TA section:

8:50-Laghi 9:55-Laghi 11:00-Basamakov 12:05-Basamakov 1:20-Jenkins 2:25-Jenkins

Problem	Points	Score
1	6	
2	6	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
11	8	
12	8	
13	8	
Total	100	

1. (6 pts) The function $\text{deg}\sin(x)$ is the ordinary sine function using degrees instead of radians. Thus, for example, $\text{deg}\sin(45) = \frac{\sqrt{2}}{2}$. Find the derivative of $\text{deg}\sin(x)$ with respect to x .

2. (6 pts) Find the following:

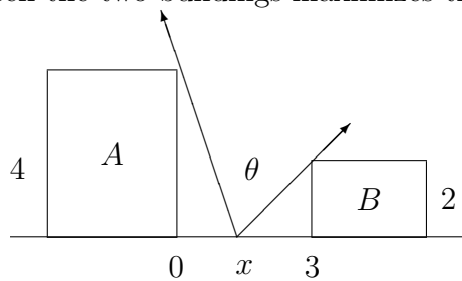
(a) $\sin^{-1}(\sin(\frac{\pi}{6})) =$

(b) $\sin^{-1}(\sin(\frac{9\pi}{8})) =$

(c) $\sin^{-1}(\sin(\frac{3\pi}{4})) =$

3. (8 pts) Prove Pythagorus's theorem.

4. (8 pts) Building A is 4 stories tall, Building B is 2 stories tall, and they are separated by a distance which is equivalent to three stories. What position (x with $0 \leq x \leq 3$) between the two buildings maximizes the viewing angle θ of sky?



5. (8 pts) Let $f(x) = e^x$. Approximate the area under this curve, above the x -axis and between the lines $x = 0$ and $x = 1$ by using the right hand rule and $\Delta x = \frac{1}{n}$. Find the limit as $n \rightarrow \infty$.

Hint: You may use the geometric series formula:

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}$$

Check that your answer is correct by using the Fundamental Theorem of Calculus to find this area.

6. (8 pts) Suppose $\int_{-2}^3 f(x) dx = 2$ and $\int_2^3 2f(x)dx = 3$. Find

$$\int_{-2}^2 (3 - 2f(x)) dx =$$

7. (8 pts) Find the minimum and maximum of $f(x) = \sqrt{3 + x^4}$ on the interval $[-1, 2]$ and use them to find lower and upper bounds L, U for the integral:

$$L \leq \int_{-1}^2 \sqrt{3 + x^4} \, dx \leq U$$

$$L =$$

$$U =$$

8. (8 pts) The hypothesis of the first part of the Fundamental Theorem of Calculus states:

Suppose that $f(x)$ is a continuous function on the interval $[a, b]$ and we define

$$I(x) = \int_a^x f(t)dt$$

for all x in $[a, b]$.

What is the conclusion of the first part of the Fundamental Theorem?

The hypothesis of the second part of the Fundamental Theorem of calculus is

Suppose $f(x)$ is a continuous function on the interval $[a, b]$ and $F(x)$ is a continuous function on $[a, b]$ such that $F'(x) = f(x)$ for all x in (a, b) .

The conclusion is that

$$\int_a^b f(t)dt = F(b) - F(a)$$

Prove it.

9. (8 pts) Find

$$\frac{d}{dx} \int_0^{x^2} \sin(t^2) dt$$

10. (8 pts) Find

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \left(1 + \frac{k}{n}\right)^5$$

11. (8 pts) Find

$$\int_0^1 \frac{x \, dx}{1 + x^4}$$

12. (8 pts) Find

$$\int_{-1}^1 x e^{x^4} dx$$

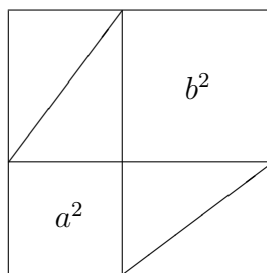
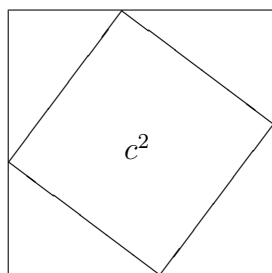
13. (8 pts) Find the constant c so that the area between the curves $y = c$ and $y = 4 - x^2$ is exactly 1.

Answers

1. $\deg\sin(x) = \sin(kx)$ where $k = \frac{2\pi}{360}$. So the derivative of $\deg\sin$ is $k \cos(kx)$.

2. The arcsin only takes values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

- (a) $\frac{\pi}{6}$
- (b) $\frac{-\pi}{8}$
- (c) $\frac{\pi}{4}$



3.

4. $6 - \frac{1}{2}\sqrt{104}$ stories to the right of A. To maximize θ minimize the sum of the other two angles to the left and right of θ .

$$f(x) = \cot^{-1}\left(\frac{x}{4}\right) + \cot^{-1}\left(\frac{3-x}{2}\right)$$

$$f'(x) = -\frac{4}{16+x^2} + \frac{2}{4+(3-x)^2}$$

Setting $f'(x) = 0$ gives us:

$$x^2 - 12x + 10 = 0$$

and so $x = 6 - \frac{1}{2}\sqrt{104}$ (the other root is outside the interval $[0, 3]$. Since $f'(0) < 0$ and $f'(3) > 0$, it must that the minimum of f (and hence maximum of θ) occurs in the interior of $[0, 3]$.

5.

$$\begin{aligned} \sum_{k=1}^n f(x_k) \Delta x &= \sum_{k=1}^n e^{\frac{k}{n}} \frac{1}{n} \\ \sum_{k=1}^n e^{\frac{k}{n}} \frac{1}{n} &= \left(\frac{1}{n}\right) \left((e^{\frac{1}{n}}) + (e^{\frac{1}{n}})^2 + (e^{\frac{1}{n}})^3 + \cdots + (e^{\frac{1}{n}})^n \right) \\ &= \left(\frac{1}{n}\right) (e^{\frac{1}{n}}) (1 + (e^{\frac{1}{n}})^1 + (e^{\frac{1}{n}})^1 + \cdots + (e^{\frac{1}{n}})^{n-1}) \end{aligned}$$

using the geometric series formula with $x = (e^{\frac{1}{n}})$ we get

$$\begin{aligned} &= \left(\frac{1}{n}\right)(e^{\frac{1}{n}}) \frac{1 - e}{1 - e^{\frac{1}{n}}} \\ &= (e^{\frac{1}{n}})(1 - e) \frac{\frac{1}{n}}{1 - e^{\frac{1}{n}}} \end{aligned}$$

$\lim_{n \rightarrow \infty} e^{\frac{1}{n}} = e^0 = 1$. Using L'Hopital's rule:

$$\lim_{n \rightarrow \infty} \frac{(\frac{1}{n})}{1 - e^{\frac{1}{n}}} = \lim_{x \rightarrow 0^+} \frac{x}{1 - e^x} = \lim_{x \rightarrow 0^+} \frac{1}{-e^x} = -1$$

so the answer is $e - 1 = \int_0^1 e^x dx = e^x \Big|_0^1$

6. 11

7. $m = \sqrt{3}$, $M = \sqrt{19}$, $L = 3\sqrt{3}$, $U = 3\sqrt{19}$

8. The conclusion of the first part is that

$I'(x) = f(x)$ for every $x \in [a, b]$.

The proof the second part is:

By the first part we know that $I'(x) = f(x) = F'(x)$ for every $x \in [a, b]$. Since two functions with the same derivative must differ by a constant, we know there exists a constant C such that $F(x) = I(x) + C$ for all $x \in [a, b]$. Hence

$$\begin{aligned} F(b) - F(a) &= (I(b) + C) - (I(a) + C) = I(b) - I(a) = \\ &= \int_a^b f(t) dt - \int_a^a f(t) dt = \int_a^b f(t) dt - 0 = \int_a^b f(t) dt \end{aligned}$$

9. $2x \sin(x^4)$

$$\begin{aligned} \frac{d}{dx} \int_0^{x^2} \sin(t^2) dt &= \frac{d}{dx} \int_0^u \sin(t^2) dt = \frac{d}{dx} I(u) = \left(\frac{d}{du} I(u)\right) \left(\frac{du}{dx}\right) = \\ &= \left(\frac{d}{du} \int_0^u \sin(t^2) dt\right) \left(\frac{du}{dx}\right) = (\sin(u^2))(2x) = (\sin(x^4))(2x) \end{aligned}$$

10. $\frac{2^6-1}{6}$. This is the same as the integral

$$\int_1^2 x^5 dx$$

11. $\frac{\pi}{8}$. Substitute $u = x^2$ and then integrate using the arctan.

12. 0. This is because $f(-x) = -f(x)$ implies that $\int_{-a}^a f(x)dx = 0$. This problem can also be worked by substituting $u = x^2$ and getting an integral $\int_1^1$ which is zero.

13. $c = 4 - (3/4)^{2/3}$ Let (a, c) be on both curves. Hence $4 - a^2 = c$. Then the area is given by

$$\int_{-a}^a ((4 - x^2) - c) dx = \int_{-a}^a ((4 - x^2) - (4 - a^2)) dx = \int_{-a}^a (a^2 - x^2) dx$$

Since this integrand is even ($f(-x) = f(x)$) we have that

$$\int_{-a}^a (a^2 - x^2) dx = 2 \int_0^a (a^2 - x^2) dx = 2 \left(a^2 x - \frac{x^3}{3} \right) \Big|_0^a = 2 \left(a^3 - \frac{a^3}{3} \right) = \frac{4}{3} a^3$$

We want this integral to equal 1 so we must have

$$\frac{4}{3} a^3 = 1$$

and so

$$a = \left(\frac{3}{4} \right)^{\frac{1}{3}}$$

and

$$c = 4 - a^2 = 4 - (3/4)^{2/3}$$